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Multislab Multigroup Transport Theory with Lth Order Anisotropic Scattering

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The special case of a triangular transfer matrix relevant to multigroup transport theory in multislabs with Lth order anisotropic scattering is discussed, and the F_N method is used to establish accurate numerical results for a test problem.

I. INTRODUCTION

In a recent article [1], which we hereafter refer to as I, we developed a method for solving, in plane geometry, the multigroup transport equation for a triangular transfer matrix that includes the effects of Lth order anisotropic scattering, and accurate numerical results were reported for the case of a single slab. Now in order to solve a larger class of practical problems, we extend the analysis given in I to include the case of multilayer systems. Because the necessary generalizations to the theory given in I can be readily visualized, we abbreviate here the presentation of our development. Of course, we assume that I is available and we use, with the inclusion of I in the equation numbers, the results developed there.

We consider the following multigroup transport equation to be applicable in each of R regions, $z \in [z_{r-1}, z_r]$, $r = 1, 2,..., R$, and for each group $i = 1, 2,..., M$:

$$\mu \frac{\partial}{\partial z} \psi_i(z, \mu) + \sigma_{i,r} \psi_i(z, \mu) = \frac{1}{2} \sum_{j=1}^{i} \sum_{l=0}^{L} \sigma_{ij}^r(l) P_l(\mu) \phi_{j,l}(z). \tag{1}$$

Here $\sigma_{i,r}$ is the total cross section for group i and region r and $\sigma_{ij}^r(l) = \sigma_{ij}^r \beta_{ij}^r(l)$, with $\beta_{ij}^r(0) = 1$, denote coefficients in Legendre expansions of the transfer cross sections. As usual, $\psi_i(z, \mu)$ represents the angular flux in the ith group and

$$\phi_{j,l}(z) = \int_{-1}^{1} \psi_j(z, \mu) P_l(\mu) \, d\mu. \tag{2}$$

We are concerned with nonmultiplying, $\sigma_{ii}^r < \sigma_{i,r}$, layers and boundary conditions of the type

$$\psi_i(z_0, \mu) = f_{i,0}(\mu), \qquad \mu > 0, \tag{3a}$$

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and

$$\psi_i(z_R, -\mu) = f_{i,R}(\mu), \qquad \mu > 0, \tag{3b}$$

where $f_{i,0}(\mu)$ and $f_{i,R}(\mu)$ are considered specified.

II. Analysis

We note that a generalization of the technique applied in I could be used here to derive a system of singular integral equations and constraints for each group, involving only the boundary data and boundary and interface angular fluxes established for previous groups. By using the F_N method in a manner similar to that reported previously for one-speed problems by Devaux *et al.* [2], we could reduce the problem of finding the unknown boundary and interface angular fluxes to one of solving a system of linear algebraic equations for each group. Since the size of each of these M systems would be, in the present case, $2(N+1)R$, it is clear that for large R the solution of very large systems would be required. We thus prefer to attack the problem in an alternative way: we consider one slab at a time, solve, for each group, R systems of $2(N+1)$ linear algebraic equations, and iterate these solutions on adjacent slabs until convergence is achieved.

We consider, for $r = 1, 2, \ldots, R$, the problem defined by Eq. (1) and boundary conditions written formally as

$$\psi_i(z_{r-1}, \mu) = L_{i,r}(\mu), \qquad \mu > 0, \tag{4a}$$

and

$$\psi_i(z_r, -\mu) = R_{i,r}(\mu), \qquad \mu > 0. \tag{4b}$$

Of course, only

$$L_{i,1}(\mu) = f_{i,0}(\mu), \qquad \mu > 0, \tag{5a}$$

and

$$R_{i,R}(\mu) = f_{i,R}(\mu), \qquad \mu > 0, \tag{5b}$$

are presently known. We now follow the analysis of I to derive, for the ith group, the system of singular integral equations and constraints given by Eqs. (I-42) and (I-44), where an index r should be included when defining quantities that depend on the particular region being considered. We let $\Delta_r = z_r - z_{r-1}$, $\Delta_{i,r} = \sigma_{i,r}\Delta_r$, and use the approximations

$$\psi_i(z_{r-1}, -\mu) = R_{i,r}(\mu)\exp(-\Delta_{i,r}/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha}^r P_\alpha(2\mu - 1) \tag{6a}$$

and

$$\psi_i(z_r, \mu) = L_{i,r}(\mu) \exp(-\Delta_{i,r}/\mu) + \sum_{\alpha=0}^{N} b^r_{i,\alpha} P_\alpha(2\mu - 1), \tag{6b}$$

for $\mu > 0$, to deduce from Eqs. (I-42) and (I-44) that

$$\sum_{\alpha=0}^{N} [a^r_{i,\alpha} B^r_{i,\alpha}(\xi) + c_{i,r} \exp(-\Delta_{i,r}/\xi)\, b^r_{i,\alpha} A^r_{i,\alpha}(\xi)] = c_{i,r} I_{i,r}(\xi) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi) \tag{7a}$$

and

$$\sum_{\alpha=0}^{N} [b^r_{i,\alpha} B^r_{i,\alpha}(\xi) + c_{i,r} \exp(-\Delta_{i,r}/\xi)\, a^r_{i,\alpha} A^r_{i,\alpha}(\xi)] = c_{i,r} J_{i,r}(\xi) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi), \tag{7b}$$

for all $\xi \in P_{i,r} = \{\nu^r_{i,m}\} \cup [0, 1]$, where $\nu^r_{i,m}$, $m = 0, 1, 2,..., \kappa_{i,r} - 1$, denote the positive discrete eigenvalues relevant to group i and region r. All quantities in Eqs. (7) are defined precisely in the same way as in I (except for the inclusion of an index r as previously noted). If we now consider Eqs. (7) at $N + 1$ values of $\xi \in P_{i,r}$, say $\xi^r_{i,\beta}$, we obtain a system of linear algebraic equations which can be written in matrix notation as

$$\mathbf{C}_{i,r} \mathbf{X}_{i,r} = \mathbf{K}_{i,r}, \tag{8}$$

where

$$\mathbf{C}_{i,r} = \begin{vmatrix} \mathbf{B}_{i,r} & \mathbf{A}_{i,r} \\ \mathbf{A}_{i,r} & \mathbf{B}_{i,r} \end{vmatrix}, \tag{9}$$

$$\mathbf{B}_{i,r} = \begin{vmatrix} B^r_{i,0}(\xi^r_{i,0}) & B^r_{i,1}(\xi^r_{i,0}) & \cdots & B^r_{i,N}(\xi^r_{i,0}) \\ B^r_{i,0}(\xi^r_{i,1}) & B^r_{i,1}(\xi^r_{i,1}) & \cdots & B^r_{i,N}(\xi^r_{i,1}) \\ \vdots & \vdots & & \vdots \\ B^r_{i,0}(\xi^r_{i,N}) & B^r_{i,1}(\xi^r_{i,N}) & \cdots & B^r_{i,N}(\xi^r_{i,N}) \end{vmatrix}, \tag{10a}$$

$$\mathbf{A}_{i,r} = \mathbf{E}_{i,r} \begin{vmatrix} A^r_{i,0}(\xi^r_{i,0}) & A^r_{i,1}(\xi^r_{i,0}) & \cdots & A^r_{i,N}(\xi^r_{i,0}) \\ A^r_{i,0}(\xi^r_{i,1}) & A^r_{i,1}(\xi^r_{i,1}) & \cdots & A^r_{i,N}(\xi^r_{i,1}) \\ \vdots & \vdots & & \vdots \\ A^r_{i,0}(\xi^r_{i,N}) & A^r_{i,1}(\xi^r_{i,N}) & \cdots & A^r_{i,N}(\xi^r_{i,N}) \end{vmatrix}, \tag{10b}$$

$$\mathbf{E}_{i,r} = c_{i,r}\{\exp(-\Delta_{i,r}/\xi^r_{i,0})\, \mathbf{k}_0 + \exp(-\Delta_{i,r}/\xi^r_{i,1})\, \mathbf{k}_1 + \cdots + \exp(-\Delta_{i,r}/\xi^r_{i,N})\, \mathbf{k}_N\}, \tag{11}$$

$\mathbf{k}_j$ is, in general, a square matrix of order $N+1$ whose elements are all zero, except for unity in the $(j+1)$th diagonal position,

$$\mathbf{X}_{i,r} = \begin{vmatrix} a^r_{i,0} \\ a^r_{i,1} \\ \vdots \\ a^r_{i,N} \\ b^r_{i,0} \\ b^r_{i,1} \\ \vdots \\ b^r_{i,N} \end{vmatrix} \tag{12}$$

and

$$\mathbf{K}_{i,r} = \begin{vmatrix} c_{i,r} I_{i,r}(\xi^r_{i,0}) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi^r_{i,0}) \\ c_{i,r} I_{i,r}(\xi^r_{i,1}) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi^r_{i,1}) \\ \vdots \\ c_{i,r} I_{i,r}(\xi^r_{i,N}) + \sum_{j=1}^{i-1} \sigma^r_{ij} I^r_{ij}(\xi^r_{i,N}) \\ c_{i,r} J_{i,r}(\xi^r_{i,0}) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi^r_{i,0}) \\ c_{i,r} J_{i,r}(\xi^r_{i,1}) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi^r_{i,1}) \\ \vdots \\ c_{i,r} J_{i,r}(\xi^r_{i,N}) + \sum_{j=1}^{i-1} \sigma^r_{ij} J^r_{ij}(\xi^r_{i,N}) \end{vmatrix} \tag{13}$$

If $\mathbf{C}^{-1}_{i,r}$ exists, Eq. (8) yields

$$\mathbf{X}_{i,r} = \mathbf{C}^{-1}_{i,r} \mathbf{K}_{i,r}, \qquad r = 1, 2, \ldots, R, \tag{14}$$

where $\mathbf{K}_{i,r}$ as given by Eq. (13), is not completely known because, as can be seen from Eqs. (I-57), $L_{i,r}(\mu)$ and $R_{i,r}(\mu)$ are required to compute $I_{i,r}(\xi)$ and $J_{i,r}(\xi)$ and, as discussed before, $L_{i,r}(\mu)$ and $R_{i,r}(\mu)$ are only formal representations as yet undetermined, with the exceptions of $L_{i,1}(\mu)$ and $R_{i,R}(\mu)$. Therefore, only the summation terms corresponding to down-scattering contributions from previous groups are supposed known in the right-hand side of Eq. (13). From the discussion, it is clear that an iterative solution can be used to establish $\mathbf{X}_{i,r}$ and consequently the emerging angular fluxes, as given by Eqs. (6). In Section III we illustrate such an iterative solution for a specific problem.

III. A Test Problem

We now consider a 20-group, 5-region albedo problem with a 10th order Legendre expansion of the scattering law. A 20-cm-thick slab has an isotropic incident distribution of radiation only in the first group and only on the surface at $z = z_0$, i.e., for $\mu > 0$

$$L_{i,1}(\mu) = \delta_{i,1} \tag{15a}$$

and

$$R_{i,R}(\mu) = 0. \tag{15b}$$

In addition, the thickness of each layer is specified by $\Delta_r = (r+1)$ cm, $r = 1, 2, \ldots, 5$. We use the fictitious cross-section set that follows to facilitate the data handling. We define, for $i = 1, 2, \ldots, 20$ and $r = 1, 2, \ldots, 5$,

$$\sigma_i^r = \left(\frac{r+20}{21}\right)^5 \left[\left(\frac{i}{10}\right) - 0.15\delta_{i,5} - 0.15\delta_{i,10}\right] \tag{16}$$

and

$$\sigma_{ij}^r(l) = (2l+1)\left(\frac{r+20}{21}\right)\left[\frac{j}{100(i-j+1)}\right](g_{ij})^l,$$

$$j = 1, 2, \ldots, i \qquad \text{and} \qquad l = 0, 1, \ldots, 10, \tag{17}$$

where

$$g_{ij} = 0.7 - (i+j)/200. \tag{18}$$

We note that the cross-section set proposed here is a simple modification of that used in I. The same collocation scheme is also used here, i.e.,

$$\xi_{i,\beta}^r = \nu_{i,\beta}^r, \qquad \beta = 0, 1, 2, \ldots, \kappa_{i,r} - 1, \tag{19}$$

and

$$\xi_{i,\beta}^r = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{2\beta - 2\kappa_{i,r} + 1}{2(N+1-\kappa_{i,r})}\pi\right],$$

$$\beta = \kappa_{i,r}, \quad \kappa_{i,r} + 1, \ldots, N. \tag{20}$$

Based on computations which closely followed the technique discussed by Siewert [3], we have concluded that there is only one pair of discrete eigenvalues relevant to each group in each region of the considered problem; we list in Table I the positive eigenvalues $\nu_{i,0}^r$ for $i = 1, 2, \ldots, 20$ and $r = 1, 2, \ldots, 5$.

We now discuss our iterative procedure to determine $\mathbf{X}_{i,r}$. Initially, for $r = 1$, we

TABLE I

The Positive Discrete Eigenvalues $\nu_{i,0}^{r}$

i	r = 1	r = 2	r = 3	r = 4	r = 5
1	1.014675230187	1.009627226459	1.006235579588	1.003967210867	1.002466094633
2	1.013664030621	1.008862139834	1.005664560862	1.003548929674	1.002166992837
3	1.012702645157	1.008140817000	1.005131686945	1.003163422111	1.001895462297
4	1.011789497866	1.007461768696	1.004635489487	1.002809198427	1.001649975857
5	1.024569285561	1.016300706575	1.010733319846	1.006980415171	1.004462235734
6	1.010101983620	1.006224775612	1.003747380593	1.002188702349	1.001231062637
7	1.009324801731	1.005664061769	1.003352641311	1.001919496742	1.001054616960
8	1.008590208601	1.005140068079	1.002988911056	1.001675703293	1.000898180681
9	1.007896906406	1.004651465922	1.002654795130	1.001455862714	1.000760271267
10	1.011201112487	1.006780184327	1.004002486528	1.002286943006	1.001254754508
11	1.006629121797	1.003775170108	1.002069829189	1.001082218593	1.000534201606
12	1.006052161369	1.003384852919	1.001816186142	1.000925519124	1.000443194621
13	1.005511523631	1.003024672419	1.001586570418	1.000786988284	1.000365036116
14	1.005005991723	1.002693310408	1.001379580893	1.000665212080	1.000298406573
15	1.004534348940	1.002389442258	1.001193817737	1.000558800707	1.000242042897
16	1.004095374943	1.002111736239	1.001027885874	1.000466395680	1.000194746410
17	1.003687842667	1.001858853819	1.000880399460	1.000386677447	1.000155390132
18	1.003310515972	1.001629450936	1.000749987263	1.000318373189	1.000122925006
19	1.002962148042	1.001422180241	1.000635298816	1.000260264497	1.000096384752
20	1.002641480584	1.001235694282	1.000535011145	1.000211194616	1.000074889135

use Eq. (15a) and, as a first guess, we set $R_{i,1}(\mu) = 0$ to compute, from Eq. (13), a crude approximation to $\mathbf{K}_{i,1}$ which we denote by $\mathbf{K}_{i,1}^{(0)}$. Accordingly, from Eq. (14), we compute $\mathbf{X}_{i,1}^{(0)}$, i.e., approximate values for $\{a_{i,\alpha}^{1}\}$ and $\{b_{i,\alpha}^{1}\}$. We now note that Eqs. (4a) and (6b) yield, for $r = 2, 3, ..., R$,

$$L_{i,r}(\mu) = L_{i,r-1}(\mu) \exp(-\Delta_{i,r-1}/\mu) + \sum_{\alpha=0}^{N} b_{i,\alpha}^{r-1} P_{\alpha}(2\mu - 1), \qquad \mu > 0, \tag{21a}$$

which together with $R_{i,r}(\mu) = 0$ can be used successively to compute approximations $\mathbf{K}_{i,r}^{(0)}$ and $\mathbf{X}_{i,r}^{(0)}$ for $r = 2, 3, ..., R$. Likewise, Eqs. (4b) and (6a) yield, for $r = 1, 2, ..., R-1$,

$$R_{i,r}(\mu) = R_{i,r+1}(\mu) \exp(-\Delta_{i,r+1}/\mu) + \sum_{\alpha=0}^{N} a_{i,\alpha}^{r+1} P_{\alpha}(2\mu - 1), \qquad \mu > 0. \tag{21b}$$

The approximations for $\{a_{i,\alpha}^{r}\}$ and $\{b_{i,\alpha}^{r}\}$, $r = 1, 2, ..., R$, found in the previous step can

TABLE II

The Exit Angular Fluxes $\psi_i(z_0, -\mu)$ for $i = 1$ to 10

μ	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8	i = 9	i = 10
0	5.0856(-2)	1.2986(-2)	5.8781(-3)	3.3590(-3)	3.1468(-3)	1.5425(-3)	1.1443(-3)	8.8437(-4)	7.0497(-4)	6.8124(-4)
0.1	2.7785(-2)	8.6312(-3)	4.2658(-3)	2.5715(-3)	2.4585(-3)	1.2658(-3)	9.5926(-4)	7.5467(-4)	6.1074(-4)	5.9839(-4)
0.2	1.7783(-2)	6.1190(-3)	3.1971(-3)	1.9979(-3)	1.9352(-3)	1.0337(-3)	7.9559(-4)	6.3435(-4)	5.1942(-4)	5.1387(-4)
0.3	1.2213(-2)	4.4866(-3)	2.4404(-3)	1.5695(-3)	1.5346(-3)	8.4721(-4)	6.6099(-4)	5.3325(-4)	4.4115(-4)	4.4002(-4)
0.4	8.8379(-3)	3.4029(-3)	1.9074(-3)	1.2545(-3)	1.2351(-3)	7.0147(-4)	5.5382(-4)	4.5146(-4)	3.7693(-4)	3.7844(-4)
0.5	6.6435(-3)	2.6534(-3)	1.5233(-3)	1.0202(-3)	1.0094(-3)	5.8743(-4)	4.6861(-4)	3.8553(-4)	3.2454(-4)	3.2753(-4)
0.6	5.1352(-3)	2.1121(-3)	1.2366(-3)	8.4089(-4)	8.3517(-4)	4.9644(-4)	3.9966(-4)	3.3154(-4)	2.8118(-4)	2.8501(-4)
0.7	4.0672(-3)	1.7123(-3)	1.0186(-3)	7.0136(-4)	6.9883(-4)	4.2292(-4)	3.4321(-4)	2.8682(-4)	2.4490(-4)	2.4920(-4)
0.8	3.2968(-3)	1.4145(-3)	8.5238(-4)	5.9298(-4)	5.9203(-4)	3.6388(-4)	2.9734(-4)	2.5006(-4)	2.1479(-4)	2.1927(-4)
0.9	2.7135(-3)	1.1849(-3)	7.2266(-4)	5.0750(-4)	5.0707(-4)	3.1630(-4)	2.6004(-4)	2.1993(-4)	1.8990(-4)	1.9432(-4)
1	2.2726(-3)	1.0048(-3)	6.1816(-4)	4.3720(-4)	4.3734(-4)	2.7596(-4)	2.2808(-4)	1.9388(-4)	1.6821(-4)	1.7260(-4)

TABLE III

The Exit Angular Fluxes $\psi_i(z_0, -\mu)$ for $i = 11$ to 20

μ	i = 11	i = 12	i = 13	i = 14	i = 15	i = 16	i = 17	i = 18	i = 19	i = 20
0	4.8119(-4)	4.0701(-4)	3.4911(-4)	3.0296(-4)	2.6556(-4)	2.3480(-4)	2.0920(-4)	1.8765(-4)	1.6933(-4)	1.5362(-4)
0.1	4.2784(-4)	3.6539(-4)	3.1617(-4)	2.7657(-4)	2.4420(-4)	2.1736(-4)	1.9485(-4)	1.7576(-4)	1.5944(-4)	1.4536(-4)
0.2	3.7165(-4)	3.1996(-4)	2.7892(-4)	2.4568(-4)	2.1831(-4)	1.9547(-4)	1.7620(-4)	1.5977(-4)	1.4564(-4)	1.3339(-4)
0.3	3.2164(-4)	2.7888(-4)	2.4475(-4)	2.1692(-4)	1.9388(-4)	1.7455(-4)	1.5815(-4)	1.4410(-4)	1.3195(-4)	1.2137(-4)
0.4	2.7946(-4)	2.4386(-4)	2.1530(-4)	1.9189(-4)	1.7241(-4)	1.5598(-4)	1.4197(-4)	1.2992(-4)	1.1946(-4)	1.1032(-4)
0.5	2.4428(-4)	2.1441(-4)	1.9034(-4)	1.7051(-4)	1.5393(-4)	1.3988(-4)	1.2786(-4)	1.1747(-4)	1.0842(-4)	1.0048(-4)
0.6	2.1461(-4)	1.8941(-4)	1.6901(-4)	1.5213(-4)	1.3795(-4)	1.2590(-4)	1.1554(-4)	1.0655(-4)	9.8688(-5)	9.1767(-5)
0.7	1.8934(-4)	1.6797(-4)	1.5061(-4)	1.3620(-4)	1.2404(-4)	1.1366(-4)	1.0470(-4)	9.6908(-5)	9.0067(-5)	8.4023(-5)
0.8	1.6796(-4)	1.4970(-4)	1.3484(-4)	1.2245(-4)	1.1197(-4)	1.0299(-4)	9.5223(-5)	8.8435(-5)	8.2461(-5)	7.7165(-5)
0.9	1.4999(-4)	1.3426(-4)	1.2141(-4)	1.1069(-4)	1.0159(-4)	9.3768(-5)	8.6980(-5)	8.1036(-5)	7.5790(-5)	7.1127(-5)
1	1.3412(-4)	1.2054(-4)	1.0944(-4)	1.0015(-4)	9.2254(-5)	8.5449(-5)	7.9528(-5)	7.4329(-5)	6.9729(-5)	6.5630(-5)

TABLE IV

The Exit Angular Fluxes $\psi_i(z_5, \mu)$ for $i = 1$ to 10

μ	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8	i = 9	i = 10
0	6.3366(-5)	1.7999(-5)	8.6499(-6)	5.1625(-6)	5.1734(-6)	2.5806(-6)	1.9573(-6)	1.5457(-6)	1.2575(-6)	1.2485(-6)
0.1	8.7378(-5)	2.3228(-5)	1.0926(-5)	6.4481(-6)	6.5023(-6)	3.1828(-6)	2.4032(-6)	1.8906(-6)	1.5329(-6)	1.5226(-6)
0.2	1.2533(-4)	3.0390(-5)	1.3902(-5)	8.0858(-6)	8.1596(-6)	3.9235(-6)	2.9460(-6)	2.3067(-6)	1.8626(-6)	1.8463(-6)
0.3	1.9871(-4)	4.0727(-5)	1.8012(-5)	1.0296(-5)	1.0379(-5)	4.8942(-6)	3.6513(-6)	2.8433(-6)	2.2848(-6)	2.2582(-6)
0.4	4.4550(-4)	5.6122(-5)	2.3833(-5)	1.3357(-5)	1.3434(-5)	6.2000(-6)	4.5920(-6)	3.5536(-6)	2.8401(-6)	2.7967(-6)
0.5	1.3050(-3)	7.9503(-5)	3.2169(-5)	1.7639(-5)	1.7691(-5)	7.9782(-6)	5.8625(-6)	4.5060(-6)	3.5797(-6)	3.5104(-6)
0.6	3.4157(-3)	1.1508(-4)	4.4073(-5)	2.3613(-5)	2.3616(-5)	1.0398(-5)	7.5780(-6)	5.7835(-6)	4.5659(-6)	4.4577(-6)
0.7	7.3116(-3)	1.6810(-4)	6.0800(-5)	3.1819(-5)	3.1758(-5)	1.3649(-5)	9.8683(-6)	7.4795(-6)	5.8685(-6)	5.7041(-6)
0.8	1.3259(-2)	2.4387(-4)	8.3692(-5)	4.2820(-5)	4.2697(-5)	1.7926(-5)	1.2865(-5)	9.6887(-6)	7.5584(-6)	7.3170(-6)
0.9	2.1265(-2)	3.4652(-4)	1.1401(-4)	5.7130(-5)	5.6985(-5)	2.3405(-5)	1.6689(-5)	1.2498(-5)	9.7006(-6)	9.3578(-6)
1	3.1162(-2)	4.7836(-4)	1.5281(-4)	7.5179(-5)	7.5100(-5)	3.0229(-5)	2.1436(-5)	1.5976(-5)	1.2348(-5)	1.1878(-5)

TABLE V

The Exit Angular Fluxes $\psi_i(z_5, \mu)$ for $i = 11$ to 20

μ	i = 11	i = 12	i = 13	i = 14	i = 15	i = 16	i = 17	i = 18	i = 19	i = 20
0	8.9495(-7)	7.6920(-7)	6.7021(-7)	5.9049(-7)	5.2518(-7)	4.7090(-7)	4.2522(-7)	3.8636(-7)	3.5299(-7)	3.2410(-7)
0.1	1.0844(-6)	9.2949(-7)	8.0777(-7)	7.0992(-7)	6.2989(-7)	5.6347(-7)	5.0765(-7)	4.6025(-7)	4.1959(-7)	3.8444(-7)
0.2	1.3080(-6)	1.1176(-6)	9.6841(-7)	8.4872(-7)	7.5102(-7)	6.7009(-7)	6.0222(-7)	5.4467(-7)	4.9540(-7)	4.5285(-7)
0.3	1.5907(-6)	1.3544(-6)	1.1696(-6)	1.0218(-6)	9.0142(-7)	8.0194(-7)	7.1870(-7)	6.4826(-7)	5.8808(-7)	5.3621(-7)
0.4	1.9579(-6)	1.6603(-6)	1.4283(-6)	1.2433(-6)	1.0930(-6)	9.6920(-7)	8.6585(-7)	7.7861(-7)	7.0426(-7)	6.4032(-7)
0.5	2.4410(-6)	2.0608(-6)	1.7654(-6)	1.5305(-6)	1.3404(-6)	1.1843(-6)	1.0543(-6)	9.4486(-7)	8.5185(-7)	7.7208(-7)
0.6	3.0781(-6)	2.5865(-6)	2.2058(-6)	1.9042(-6)	1.6610(-6)	1.4618(-6)	1.2965(-6)	1.1577(-6)	1.0402(-6)	9.3958(-7)
0.7	3.9116(-6)	3.2715(-6)	2.7775(-6)	2.3875(-6)	2.0740(-6)	1.8181(-6)	1.6064(-6)	1.4293(-6)	1.2795(-6)	1.1519(-6)
0.8	4.9849(-6)	4.1508(-6)	3.5091(-6)	3.0042(-6)	2.5995(-6)	2.2701(-6)	1.9985(-6)	1.7718(-6)	1.5808(-6)	1.4183(-6)
0.9	6.3381(-6)	5.2567(-6)	4.4272(-6)	3.7763(-6)	3.2560(-6)	2.8337(-6)	2.4863(-6)	2.1972(-6)	1.9541(-6)	1.7478(-6)
1	8.0043(-6)	6.6163(-6)	5.5541(-6)	4.7226(-6)	4.0596(-6)	3.5226(-6)	3.0818(-6)	2.7157(-6)	2.4085(-6)	2.1483(-6)

TABLE VI

The Angular Fluxes $\psi_i(z,\mu)$ for $z = z_2 + \Delta_3/2$ and $i = 1$ to 10

μ	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7	i = 8	i = 9	i = 10
-1	3.6184(-4)	1.5638(-4)	9.3646(-5)	6.4533(-5)	6.6068(-5)	3.9539(-5)	3.1976(-5)	2.6682(-5)	2.2774(-5)	2.3252(-5)
-0.8	5.0171(-4)	2.0930(-4)	1.2277(-4)	8.3348(-5)	8.5646(-5)	4.9959(-5)	4.0050(-5)	3.3160(-5)	2.8108(-5)	2.8655(-5)
-0.6	7.2982(-4)	2.9089(-4)	1.6621(-4)	1.1075(-4)	1.1420(-4)	6.4587(-5)	5.1241(-5)	4.2047(-5)	3.5361(-5)	3.5967(-5)
-0.4	1.1346(-3)	4.2492(-4)	2.3438(-4)	1.5236(-4)	1.5755(-4)	8.5842(-5)	6.7294(-5)	5.4664(-5)	4.5574(-5)	4.6189(-5)
-0.2	1.9474(-3)	6.6092(-4)	3.4641(-4)	2.1809(-4)	2.2566(-4)	1.1797(-4)	9.1278(-5)	7.3334(-5)	6.0556(-5)	6.1062(-5)
0	3.7995(-3)	1.0988(-3)	5.4058(-4)	3.2815(-4)	3.3814(-4)	1.6929(-4)	1.2903(-4)	1.0233(-4)	8.3552(-5)	8.3687(-5)
0.2	2.1021(-2)	2.0301(-3)	9.0271(-4)	5.2139(-4)	5.3497(-4)	2.5349(-4)	1.8976(-4)	1.4822(-4)	1.1942(-4)	1.1866(-4)
0.4	1.2432(-1)	3.8384(-3)	1.5599(-3)	8.4639(-4)	8.7182(-4)	3.8345(-4)	2.8147(-4)	2.1636(-4)	1.7198(-4)	1.6966(-4)
0.6	2.4736(-1)	5.8053(-3)	2.4324(-3)	1.2967(-3)	1.3293(-3)	5.5908(-4)	4.0352(-4)	3.0587(-4)	2.4029(-4)	2.3615(-4)
0.8	3.5123(-1)	7.2855(-3)	3.2564(-3)	1.7745(-3)	1.7904(-3)	7.5900(-4)	5.4334(-4)	4.0840(-4)	3.1833(-4)	3.1261(-4)
1	4.3378(-1)	8.2115(-3)	3.8945(-3)	2.1939(-3)	2.1725(-3)	9.5642(-4)	6.8511(-4)	5.1381(-4)	3.9913(-4)	3.9222(-4)

TABLE VII

The Angular Fluxes $\psi_i(z,\mu)$ for $z = z_2 + \Delta_3/2$ and $i = 11$ to 20

μ	i = 11	i = 12	i = 13	i = 14	i = 15	i = 16	i = 17	i = 18	i = 19	i = 20
-1	1.7696(-5)	1.5718(-5)	1.4123(-5)	1.2805(-5)	1.1699(-5)	1.0758(-5)	9.9483(-6)	9.2458(-6)	8.6311(-6)	8.0892(-6)
-0.8	2.1572(-5)	1.9061(-5)	1.7044(-5)	1.5385(-5)	1.3996(-5)	1.2819(-5)	1.1810(-5)	1.0937(-5)	1.0175(-5)	9.5048(-6)
-0.6	2.6766(-5)	2.3519(-5)	2.0921(-5)	1.8793(-5)	1.7020(-5)	1.5522(-5)	1.4243(-5)	1.3139(-5)	1.2178(-5)	1.1336(-5)
-0.4	3.3976(-5)	2.9672(-5)	2.6246(-5)	2.3451(-5)	2.1133(-5)	1.9182(-5)	1.7522(-5)	1.6095(-5)	1.4857(-5)	1.3775(-5)
-0.2	4.4376(-5)	3.8491(-5)	3.3828(-5)	3.0044(-5)	2.6920(-5)	2.4303(-5)	2.2086(-5)	2.0188(-5)	1.8547(-5)	1.7119(-5)
0	5.9990(-5)	5.1616(-5)	4.5020(-5)	3.9700(-5)	3.5333(-5)	3.1697(-5)	2.8631(-5)	2.6020(-5)	2.3774(-5)	2.1827(-5)
0.2	8.3727(-5)	7.1365(-5)	6.1701(-5)	5.3963(-5)	4.7657(-5)	4.2440(-5)	3.8070(-5)	3.4370(-5)	3.1206(-5)	2.8477(-5)
0.4	1.1776(-4)	9.9441(-5)	8.5227(-5)	7.3932(-5)	6.4792(-5)	5.7282(-5)	5.1031(-5)	4.5769(-5)	4.1295(-5)	3.7457(-5)
0.6	1.6132(-4)	1.3517(-4)	1.1502(-4)	9.9120(-5)	8.6325(-5)	7.5871(-5)	6.7214(-5)	5.9961(-5)	5.3822(-5)	4.8578(-5)
0.8	2.1080(-4)	1.7565(-4)	1.4871(-4)	1.2754(-4)	1.1060(-4)	9.6807(-5)	8.5428(-5)	7.5928(-5)	6.7913(-5)	6.1087(-5)
1	2.6230(-4)	2.1779(-4)	1.8377(-4)	1.5713(-4)	1.3587(-4)	1.1860(-4)	1.0440(-4)	9.2568(-5)	8.2607(-5)	7.4141(-5)

now be used in Eqs. (21), and Eq. (14) can be solved successively for $r = R - 1$, $R - 2, \ldots, 2, 1$ to provide improved approximations for $\{a^r_{i,\alpha}\}$ and $\{b^r_{i,\alpha}\}$. This completes what we call the first sweep of the multislab system. Second, third, and higher order sweeps can be repeated in a similar way by using the updated approximations for $\{a^r_{i,\alpha}\}$ and $\{b^r_{i,\alpha}\}$ until the relative error in these constants for two successive sweeps is as small as desired. In our problem we have found that, in general, five or six sweeps were sufficient to achieve a relative error smaller than 10^{-10}. Once the converged values of $\{a^r_{i,\alpha}\}$ and $\{b^r_{i,\alpha}\}$ are available, appropriate versions of Eqs. (I-124) can be used to compute the angular fluxes emerging from each region, and interior fluxes can then be computed as in I. We list our converged results for $\psi_i(z_0, -\mu)$ and $\psi_i(z_5, \mu)$, $\mu > 0$, $i = 1, 2, \ldots, 20$, in Tables II–V. Converged results for the angular flux at the center of region 3, $z = z_2 + \Delta_3/2$, are reported in Tables VI and VII and for the group fluxes

$$\phi_i(z) = \int_{-1}^{1} \psi_i(z, \mu)\, d\mu \tag{22}$$

TABLE VIII

The Group Fluxes $\phi_i(z)$

i	z = z0	z = z2+Δ3/2	z = z5
1	1.0112	1.9270(-1)	6.2131(-3)
2	3.8166(-3)	5.1551(-3)	1.3399(-4)
3	2.0235(-3)	2.3027(-3)	4.7871(-5)
4	1.2877(-3)	1.2863(-3)	2.4981(-5)
5	1.2533(-3)	1.3082(-3)	2.4979(-5)
6	6.9222(-4)	5.8515(-4)	1.0738(-5)
7	5.4114(-4)	4.2893(-4)	7.7753(-6)
8	4.3784(-4)	3.2930(-4)	5.9019(-6)
9	3.6347(-4)	2.6157(-4)	4.6376(-6)
10	3.6290(-4)	2.5884(-4)	4.5126(-6)
11	2.6722(-4)	1.7926(-4)	3.1007(-6)
12	2.3262(-4)	1.5150(-4)	2.5970(-6)
13	2.0497(-4)	1.2999(-4)	2.2080(-6)
14	1.8241(-4)	1.1293(-4)	1.9006(-6)
15	1.6369(-4)	9.9125(-5)	1.6534(-6)
16	1.4796(-4)	8.7789(-5)	1.4514(-6)
17	1.3459(-4)	7.8354(-5)	1.2842(-6)
18	1.2311(-4)	7.0410(-5)	1.1443(-6)
19	1.1317(-4)	6.3655(-5)	1.0259(-6)
20	1.0449(-4)	5.7858(-5)	9.2482(-7)

TABLE IX

The Group Albedos A_i^* and the Transmission Factors B_i^*

	Present Work		DTF69	
i	A_i^*	B_i^*	A_i^*	B_i^*
1	5.8809(-3)	1.0453(-2)	5.8821(-3)	1.0439(-2)
2	2.2791(-3)	1.9993(-4)	2.2798(-3)	1.9965(-4)
3	1.2939(-3)	6.9012(-5)	1.2944(-3)	6.8915(-5)
4	8.6280(-4)	3.5393(-5)	8.6319(-4)	3.5345(-5)
5	8.5170(-4)	3.5350(-5)	8.5202(-4)	3.5300(-5)
6	4.9662(-4)	1.4899(-5)	4.9692(-4)	1.4879(-5)
7	3.9706(-4)	1.0716(-5)	3.9733(-4)	1.0702(-5)
8	3.2763(-4)	8.0863(-6)	3.2787(-4)	8.0757(-6)
9	2.7671(-4)	6.3202(-6)	2.7693(-4)	6.3119(-6)
10	2.7956(-4)	6.1271(-6)	2.7977(-4)	6.1191(-6)
11	2.0989(-4)	4.1837(-6)	2.1007(-4)	4.1782(-6)
12	1.8491(-4)	3.4892(-6)	1.8508(-4)	3.4847(-6)
13	1.6476(-4)	2.9545(-6)	1.6491(-4)	2.9506(-6)
14	1.4814(-4)	2.5332(-6)	1.4828(-4)	2.5299(-6)
15	1.3423(-4)	2.1953(-6)	1.3435(-4)	2.1924(-6)
16	1.2242(-4)	1.9200(-6)	1.2253(-4)	1.9174(-6)
17	1.1229(-4)	1.6927(-6)	1.1240(-4)	1.6904(-6)
18	1.0352(-4)	1.5029(-6)	1.0362(-4)	1.5009(-6)
19	9.5859(-5)	1.3428(-6)	9.5952(-5)	1.3409(-6)
20	8.9125(-5)	1.2064(-6)	8.9210(-5)	1.2047(-6)

in Table VIII. Finally, converged results for the group albedos

$$A_i^* = 2 \int_0^1 \mu\psi_i(z_0, -\mu)\, d\mu \tag{23a}$$

and the group transmission factors

$$B_i^* = 2 \int_0^1 \mu\psi_i(z_5, \mu)\, d\mu \tag{23b}$$

are reported in Table IX, along with those found by Renken [4] who used the code DTF69 with 75 space points and eight discrete directions [5] for each half range of μ. All our numerical results are, we believe, accurate to within ± 1 in the last digit, and the degree of agreement with the DTF69 results is essentially the same as in I.

Finally, in regard to the convergence of our method, we have found that $N = 20$

was required to establish the boundary and interface fluxes accurate to five significant figures; for the integrated quantities $\phi_l(z)$, A_l^*, and B_l^*, and interior angular fluxes $N = 15$ was sufficient to obtain five figures of accuracy.

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