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THE F_N METHOD FOR POLARIZATION STUDIES—II NUMERICAL RESULTS

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Abstract—The F_N method is used to establish numerical results basic to polarization studies in plane parallel atmospheres.

1. INTRODUCTION

In a recent paper,[1] hereafter referred to as I, the F_N method was used to establish an approximate semi-analytical solution of the equation of transfer

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau,\mu)+\mathbf{I}(\tau,\mu)=\frac{1}{2}\omega\mathbf{Q}(\mu)\int_{-1}^{1}\mathbf{Q}^T(\mu')\mathbf{I}(\tau,\mu')\mathrm{d}\mu',\quad \tau\in[0,\tau_0],\tag{1}$$

subject to the boundary conditions

$$\mathbf{I}(0,\mu)=\mathbf{F}_1(\mu),\quad \mu>0,\tag{2a}$$

and

$$\mathbf{I}(\tau_0,-\mu)=\mathbf{F}_2(\mu),\quad \mu>0.\tag{2b}$$

Here we wish to report some numerical results that show the computational merit of the method.

We use $\mathbf{I}(\tau,\mu)$, with components $I_l(\tau,\mu)$ and $I_r(\tau,\mu)$ to characterize the azimuthally symmetric component of the polarized radiation field. We also write

$$\mathbf{Q}(\mu)=\frac{3}{2}(c+2)^{-1/2}\begin{vmatrix}c\mu^2+(2/3)(1-c) & (2c)^{1/2}(1-\mu^2)\\ (1/3)(c+2) & 0\end{vmatrix}\tag{3}$$

and allow a combination of Rayleigh and isotropic scattering.[2] As reported in I, we can write

$$\mathbf{I}(\tau,\mu)=A(\eta_0)\mathbf{\Phi}(\eta_0,\mu)\mathrm{e}^{-\tau/\eta_0}+A(-\eta_0)\mathbf{\Phi}(-\eta_0,\mu)\mathrm{e}^{\tau/\eta_0}+\int_{-1}^{1}\mathbf{\Phi}(\eta,\mu)\mathbf{A}(\eta)\mathrm{e}^{-\tau/\eta}\mathrm{d}\eta\tag{4}$$

and use the full-range orthogonality properties of the functions $\mathbf{\Phi}(\xi,\mu)$ to deduce the following system of singular integral equations and constraints for the surface intensities:

$$\int_0^1\mu\mathbf{\Phi}^T(\xi,\mu)\mathbf{I}(0,-\mu)\mathrm{d}\mu+\mathrm{e}^{-\tau_0/\xi}\int_0^1\mu\mathbf{\Phi}^T(-\xi,\mu)\mathbf{I}(\tau_0,\mu)\mathrm{d}\mu=\mathbf{L}_1(\xi),\quad \xi\in P,\tag{5a}$$

and

$$\int_0^1\mu\mathbf{\Phi}^T(\xi,\mu)\mathbf{I}(\tau_0,\mu)\mathrm{d}\mu+\mathrm{e}^{-\tau_0/\xi}\int_0^1\mu\mathbf{\Phi}^T(-\xi,\mu)\mathbf{I}(0,-\mu)\mathrm{d}\mu=\mathbf{L}_2(\xi),\quad \xi\in P,\tag{5b}$$

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where the known functions are

$$\mathbf{L}_1(\xi) = \int_0^1 \mu \mathbf{\Phi}^T(-\xi, \mu)\mathbf{F}_1(\mu)\,\mathrm{d}\mu + \mathrm{e}^{-\tau_0/\xi} \int_0^1 \mu \mathbf{\Phi}^T(\xi, \mu)\mathbf{F}_2(\mu)\,\mathrm{d}\mu \tag{6a}$$

and

$$\mathbf{L}_2(\xi) = \int_0^1 \mu \mathbf{\Phi}^T(-\xi, \mu)\mathbf{F}_2(\mu)\mathrm{d}\mu + \mathrm{e}^{-\tau_0/\xi} \int_0^1 \mu \mathbf{\Phi}^T(\xi, \mu)\mathbf{F}_1(\mu)\mathrm{d}\mu; \tag{6b}$$

also, $\xi \in P \Rightarrow \xi \in \eta_0 \cup (0, 1)$. In order to be brief we do not repeat here all of the basic definitions given in I.

2. ANALYSIS

We now consider the *planetary problem* formulated by Chandrasekhar,[2] i.e. the illumination of an atmosphere of optical thickness τ_0 by a beam of polarized light. At the *ground* location, $\tau = \tau_0$, we allow reflection according to Lambert's law,[2] with an albedo λ_0, and thus for this application Eqs. (2) become

$$\mathbf{I}(0, \mu) = \frac{1}{2}\delta(\mu - \mu_0)\mathbf{F}, \quad \mu, \mu_0 > 0, \tag{7a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = \lambda_0 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \int_0^1 \mathbf{I}(\tau_0, \mu')\mu'\mathrm{d}\mu', \quad \mu > 0, \tag{7b}$$

where the constant $\mathbf{F}$ has components F_l and F_r. Equation (7b) clearly indicates that the "ground" introduces a component of unpolarized light into the atmosphere. We recall that the $\mathbf{Q}(\mu)$ given by Eq. (3) already allows (if $c \neq 1$) a combination of Rayleigh and depolarizing isotropic scattering within the atmosphere.

We now introduce the F_N method, and thus we approximate the unknown surface intensities by

$$\mathbf{I}(0, -\mu) = \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{a}_\alpha \mu^\alpha, \quad \mu > 0, \tag{8a}$$

and

$$\mathbf{I}(\tau_0, \mu) = \frac{1}{2}\delta(\mu - \mu_0)\mathrm{e}^{-\tau_0/\mu_0}\mathbf{F} + \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{b}_\alpha \mu^\alpha, \quad \mu > 0, \tag{8b}$$

where the constants $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ are to be determined. In Eqs. (8) we have included a factor $\mathbf{Q}(\mu)$ that was not used in I. If we substitute Eqs. (8) into Eqs. (5) and use the boundary conditions given by Eqs. (7), we find

$$\sum_{\alpha=0}^{N} \mathbf{\Delta}_\alpha(\xi)\mathbf{a}_\alpha + \mathrm{e}^{-\tau_0/\xi} \sum_{\alpha=0}^{N} \left\{\mathbf{\Gamma}_\alpha(\xi) - \lambda_0 \mathbf{B}_0{}^T(\xi)\mathbf{D}\left[\mathbf{Q}_0\left(\frac{1}{\alpha+2}\right) + \mathbf{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\}\mathbf{b}_\alpha = \mathbf{K}_1(\xi) \tag{9a}$$

and

$$\sum_{\alpha=0}^{N} \left\{\mathbf{\Delta}_\alpha(\xi) - \lambda_0 \mathbf{A}_0{}^T(\xi)\mathbf{D}\left[\mathbf{Q}_0\left(\frac{1}{\alpha+2}\right) + \mathbf{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\}\mathbf{b}_\alpha + \mathrm{e}^{-\tau_0/\xi} \sum_{\alpha=0}^{N} \mathbf{\Gamma}_\alpha(\xi)\mathbf{a}_\alpha = \mathbf{K}_2(\xi), \tag{9b}$$

where

$$\mathbf{D} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \tag{10}$$

$$\mathbf{A}_0(\xi) = \left[\mathbf{Q}_0 + \frac{1}{3}\mathbf{Q}_2 + \xi\left(\xi - \frac{1}{2}\right)\mathbf{Q}_2 - \xi\mathbf{Q}(\xi)\log\left(1 + \frac{1}{\xi}\right)\right]\mathbf{M}(\xi), \tag{11a}$$

$$\mathbf{B}_0(\xi) = \left[\frac{2}{\omega}\boldsymbol{\Delta}(\xi) - \mathbf{Q}_0 - \frac{1}{3}\mathbf{Q}_2 + \xi\left(\xi - \frac{1}{2}\right)\mathbf{Q}_2 - \xi\mathbf{Q}(\xi)\log\left(1 + \frac{1}{\xi}\right)\right]\mathbf{M}(\xi), \tag{11b}$$

$$\boldsymbol{\Delta}(\xi) = \mathbf{Q}^{-T}(\xi)\left[\mathbf{I} + \omega\xi^2\mathbf{Q}_2^T\left(\mathbf{Q}_0 + \frac{1}{3}\mathbf{Q}_2\right)\right], \tag{11c}$$

and the known functions are,

$$\mathbf{K}_1(\xi) = \frac{1}{2}\mu_0\left\{[1 - e^{-\tau_0[(1/\mu_0)+(1/\xi)]}]\frac{2}{\omega\xi}\boldsymbol{\Phi}^T(-\xi, \mu_0) + \lambda_0 e^{-\tau_0[(1/\mu_0)+(1/\xi)]}\mathbf{B}_0^T(\xi)\mathbf{D}\right\}\mathbf{F} \tag{12a}$$

and

$$\mathbf{K}_2(\xi) = \frac{1}{2}\mu_0\left\{[e^{-\tau_0/\xi} - e^{-\tau_0/\mu_0}]\frac{2}{\omega\xi}\boldsymbol{\Phi}^T(\xi, \mu_0) + \lambda_0 e^{-\tau_0/\mu_0}\mathbf{A}_0^T(\xi)\mathbf{D}\right\}\mathbf{F}. \tag{12b}$$

The functions $\boldsymbol{\Gamma}_\alpha(\xi)$ and $\boldsymbol{\Delta}_\alpha(\xi)$ appearing in Eqs. (9) are defined by

$$\boldsymbol{\Gamma}_\alpha(\xi) = \frac{2}{\omega\xi}\int_0^1 \mu^{\alpha+1}\boldsymbol{\Phi}^T(-\xi, \mu)\mathbf{Q}(\mu)\,d\mu \tag{13a}$$

and

$$\boldsymbol{\Delta}_\alpha(\xi) = \frac{2}{\omega\xi}\int_0^1 \mu^{\alpha+1}\boldsymbol{\Phi}^T(\xi, \mu)\mathbf{Q}(\mu)\,d\mu \tag{13b}$$

and can readily be generated from

$$\boldsymbol{\Gamma}_\alpha(\xi) = -\xi\boldsymbol{\Gamma}_{\alpha-1}(\xi) + \mathbf{M}^T(\xi)\mathbf{R}_\alpha \tag{14a}$$

with

$$\boldsymbol{\Gamma}_0(\xi) = \mathbf{M}^T(\xi)[\mathbf{R}_0 - \xi\mathbf{K}(\xi)] \tag{14b}$$

and

$$\boldsymbol{\Delta}_\alpha(\xi) = \xi\boldsymbol{\Delta}_{\alpha-1}(\xi) - \mathbf{M}^T(\xi)\mathbf{R}_\alpha \tag{15a}$$

with

$$\boldsymbol{\Delta}_0(\xi) = \mathbf{M}^T(\xi)\left[\frac{2}{\omega}\mathbf{I} - \mathbf{R}_0 - \xi\mathbf{K}(\xi)\right]. \tag{15b}$$

Here

$$\mathbf{R}_\alpha = \mathbf{Q}_0^T\mathbf{Q}_0\left(\frac{1}{\alpha+1}\right) + (\mathbf{Q}_2^T\mathbf{Q}_0 + \mathbf{Q}_0^T\mathbf{Q}_2)\left(\frac{1}{\alpha+3}\right) + \mathbf{Q}_2^T\mathbf{Q}_2\left(\frac{1}{\alpha+5}\right) \tag{16}$$

and

$$\mathbf{K}(\xi) = \mathbf{Q}_0^T\mathbf{Q}_0\log\left(1 + \frac{1}{\xi}\right) + (\mathbf{Q}_2^T\mathbf{Q}_0 + \mathbf{Q}_0^T\mathbf{Q}_2)\left[\xi^2\log\left(1 + \frac{1}{\xi}\right) + \frac{1}{2} - \xi\right]$$
$$+ \mathbf{Q}_2^T\mathbf{Q}_2\left[\xi^4\log\left(1 + \frac{1}{\xi}\right) + \frac{1}{4} - \frac{\xi}{3} + \frac{\xi^2}{2} - \xi^3\right], \tag{17}$$

where we have written

$$\mathbf{Q}(\mu) = \mathbf{Q}_0 + \mu^2\mathbf{Q}_2. \tag{18}$$

We use $\mathbf{M}(\xi) = \mathbf{I}$, $\xi \in (0, 1)$, whereas $\mathbf{M}(\eta_0)$ is a null vector of the dispersion matrix,[3] i.e.

$$\Lambda(\eta_0)\mathbf{M}(\eta_0) = \mathbf{0}, \tag{19}$$

where

$$\Lambda(z) = \mathbf{I} + z \int_{-1}^{1} \mathbf{\Psi}(x)\frac{\mathrm{d}x}{x - z} \tag{20}$$

and the characteristic matrix is

$$\mathbf{\Psi}(x) = \frac{1}{2}\omega \mathbf{Q}^T(x)\mathbf{Q}(x). \tag{21}$$

It is apparent that, once the desired constants $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ are found, the surface intensities become known, and thus other quantities of physical interest are readily available. For example, the albedo

$$A^* = 2[\mu_0(F_l + F_r)]^{-1} \int_0^1 [I_l(0, -\mu) + I_r(0, -\mu)]\mu \, \mathrm{d}\mu \tag{22a}$$

and the transmission factor

$$B^* = 2[\mu_0(F_l + F_r)]^{-1} \int_0^1 [I_l(\tau_0, \mu) + I_r(\tau_0, \mu)]\mu \, \mathrm{d}\mu \tag{22b}$$

can be expressed, in the F_N approximation, by

$$A^* = 2[\mu_0(F_l + F_r)]^{-1} \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \sum_{\alpha=0}^{N} \left[\left(\frac{1}{\alpha+2}\right)\mathbf{Q}_0 + \left(\frac{1}{\alpha+4}\right)\mathbf{Q}_2\right]\mathbf{a}_\alpha \tag{23a}$$

and

$$B^* = 2[\mu_0(F_l + F_r)]^{-1} \left\{\frac{1}{2}\mu_0(F_l + F_r)\mathrm{e}^{-\tau_0/\mu_0} + \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \sum_{\alpha=0}^{N} \left[\left(\frac{1}{\alpha+2}\right)\mathbf{Q}_0 + \left(\frac{1}{\alpha+4}\right)\mathbf{Q}_2\right]\mathbf{b}_\alpha\right\}. \tag{23b}$$

In a similar manner, the Stokes parameters[2] at the two surfaces

$$I(0, -\mu) = I_l(0, -\mu) + I_r(0, -\mu), \quad \mu > 0, \tag{24a}$$

$$I^*(\tau_0, \mu) = I_l^*(\tau_0, \mu) + I_r^*(\tau_0, \mu), \quad \mu > 0, \tag{24b}$$

$$Q(0, -\mu) = I_l(0, -\mu) - I_r(0, -\mu), \quad \mu > 0, \tag{24c}$$

and

$$Q^*(\tau_0, \mu) = I_l^*(\tau_0, \mu) - I_r^*(\tau_0, \mu), \quad \mu > 0, \tag{24d}$$

where

$$\mathbf{I}^*(\tau, \mu) = \mathbf{I}(\tau, \mu) - \frac{1}{2}\delta(\mu - \mu_0)\mathrm{e}^{-\tau/\mu}\mathbf{F}, \tag{25}$$

can be expressed in the F_N approximation as

$$I(0, -\mu) = \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{a}_\alpha \mu^\alpha, \quad \mu > 0, \tag{26a}$$

$$I^*(\tau_0, \mu) = \begin{vmatrix} 1 \\ 1 \end{vmatrix}^T \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{b}_\alpha \mu^\alpha, \quad \mu > 0, \tag{26b}$$

$$Q(0, -\mu) = \begin{vmatrix} 1 \\ -1 \end{vmatrix}^T \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{a}_\alpha \mu^\alpha, \quad \mu > 0, \tag{26c}$$

and

$$Q^*(\tau_0, \mu) = \begin{vmatrix} 1 \\ -1 \end{vmatrix}^T \mathbf{Q}(\mu) \sum_{\alpha=0}^{N} \mathbf{b}_\alpha \mu^\alpha, \quad \mu > 0. \tag{26d}$$

3. NUMERICAL RESULTS

In order to find the required constants $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$ we now consider the following set of linear algebraic equations generated by evaluating Eqs. (9) at selected values, say $\{\xi_j\}$, of $\xi \in P$:

$$\sum_{\alpha=0}^{N} \mathbf{\Delta}_\alpha(\xi_j)\mathbf{a}_\alpha + e^{-\tau_0/\xi_j} \sum_{\alpha=0}^{N} \left\{\mathbf{\Gamma}_\alpha(\xi_j) - \lambda_0 \mathbf{B}_0{}^T(\xi_j)\mathbf{D}\left[\mathbf{Q}_0\left(\frac{1}{\alpha+2}\right) + \mathbf{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\}\mathbf{b}_\alpha = \mathbf{K}_1(\xi_j) \tag{27a}$$

and

$$\sum_{\alpha=0}^{N} \left\{\mathbf{\Delta}_\alpha(\xi_j) - \lambda_0 \mathbf{A}_0{}^T(\xi_j)\mathbf{D}\left[\mathbf{Q}_0\left(\frac{1}{\alpha+2}\right) + \mathbf{Q}_2\left(\frac{1}{\alpha+4}\right)\right]\right\}\mathbf{b}_\alpha + e^{-\tau_0/\xi_j} \sum_{\alpha=0}^{N} \mathbf{\Gamma}_\alpha(\xi_j)\mathbf{a}_\alpha = \mathbf{K}_2(\xi_j). \tag{27b}$$

In I a particular scheme was suggested for selecting the points $\{\xi_j\}$, and a restriction was placed on the form of $\mathbf{a}_0$ and $\mathbf{b}_0$. We have tried the scheme suggested in I and have concluded that the restriction placed on $\mathbf{a}_0$ and $\mathbf{b}_0$ impedes the numerical accuracy of the method. In addition, the point $\xi = 1$ cannot be used as suggested in I, since $\mathbf{Q}(1)$ is singular. Here we use the approximations given by Eqs. (8). We obtain a simple (and effective) scheme for choosing the points by using, for the F_0 approximation, $\xi_0 = \eta_0$ in Eqs. (27) to establish two scalar equations, while for $\xi_1 = 1$ we use only two of the four scalar equations obtained from Eqs. (27); in particular, we use for $\xi_1 = 1$ the upper (scalar) equations in Eqs. (27). In addition, we define, for higher order F_N approximations, $\xi_j = (2j-3)/(2N)$, $j = 2,3,4,\ldots N+1$, and use each of the four resulting scalar equations, for each ξ_j, $j > 1$, from Eqs. (27). Thus, in general, we must solve a system of $4(N+1)$ linear algebraic equations to establish the desired constants $\{\mathbf{a}_\alpha\}$ and $\{\mathbf{b}_\alpha\}$.

From a computational point-of-view, the first result we wish to establish is η_0. We thus use the explicit expression

$$\eta_0 = \left[(1-\omega)\left(1 - \frac{7}{10}\omega c\right)\right]^{-1/2} \exp\left[-\frac{1}{\pi}\int_0^1 \theta(t)\frac{dt}{t}\right], \tag{28}$$

where $\theta(t)$ is known.[4] The result can then be refined by solving iteratively det $\Lambda(z) = 0$. Once η_0 was established, we solved the system of linear algebraic equations given by Eqs. (27), with $F_l = F_r = 1/2$, for the data cases given in Table 1. In Tables 2 and 3, we report the albedo and transmission factor as computed from various orders of the F_N approximation. We also list "Exact" results deduced from "converged" F_N computations and confirmed, for the half space, with a previous work.[3] Tables 4–6 show the "converged" F_N results for the Stokes parameters for data cases 1, 3, and 5. For the half space, the results of Table 6 agree with the exact analysis of Bond and Siewert.[3] For the finite atmospheres, our "converged" results for $I(0, -\mu)$ and $Q(0, -\mu)$ shown in Tables 4 and 5 agree to the given accuracy with the work of Kawabata[5] and Hansen.[6]

We note that our F_N calculations have generally "converged" to five significant figures, except near $\mu = 0$ and $\mu = 1$, for the angular-dependent Stokes parameters. The method

Table 1. Data.

Case	ω	c	λ_0	τ_0	μ_0
1	0.9	1.0	0.0	1.0	1.0
2	0.9	1.0	0.1	1.0	1.0
3	0.9	0.8	0.1	5.0	1.0
4	0.9	0.8	0.2	5.0	1.0
5	0.9	0.8	—	∞	1.0

Table 2. The albedo A^*.

Case	F_3	F_5	F_7	"Exact"
1	0.27016	0.27022	0.27023	0.27023
2	0.29950	0.29956	0.29957	0.29957
3	0.41753	0.41754	0.41754	0.41754
4	0.41802	0.41803	0.41803	0.41803
5	0.41960	0.41961	0.41961	0.41961

Table 3. The transmission factor B^*.

Case	F_3	F_5	F_7	"Exact"
1	0.59591	0.59618	0.59617	0.59617
2	0.61772	0.61800	0.61800	0.61799
3	0.082955	0.082958	0.082958	0.082958
4	0.087331	0.087334	0.087334	0.087334
5	—	—	—	0.0

Table 4. "Converged" results for case 1.

μ	$I(0,-\mu)$	$-Q(0,-\mu)$	$I^*(\tau_0,\mu)$	$-Q^*(\tau_0,\mu)$
0.02	0.2812	0.2184	0.1549	0.1003
0.06	0.2856	0.2155	0.1671	0.1062
0.10	0.2875	0.2105	0.1776	0.1109
0.16	0.2879	0.2006	0.1919	0.1165
0.20	0.2869	0.1928	0.1998	0.1187
0.28	0.2829	0.1749	0.2111	0.1183
0.32	0.2803	0.1652	0.2148	0.1159
0.40	0.2751	0.1450	0.2198	0.1078
0.52	0.2688	0.1142	0.2241	0.09020
0.64	0.2655	0.08401	0.2277	0.06914
0.72	0.2651	0.06449	0.2305	0.05417
0.84	0.2667	0.03616	0.2358	0.03110
0.92	0.2692	0.01786	0.2401	0.01556
0.96	0.2708	0.00888	0.2425	0.00778
0.98	0.2717	0.0044	0.2438	0.0039
1.00	0.2726	0.0	0.2450	0.0

Table 5. "Converged" results for case 3.

μ	$I(0,-\mu)$	$-Q(0,-\mu)$	$I^*(\tau_0,\mu)$	$-Q^*(\tau_0,\mu)\times 100$
0.02	0.3624	0.1660	0.03444	0.4432
0.06	0.3730	0.1617	0.03686	0.4457
0.10	0.3801	0.1567	0.03912	0.4499
0.16	0.3876	0.1484	0.04242	0.4568
0.20	0.3913	0.1424	0.04462	0.4613
0.28	0.3971	0.1300	0.04911	0.4685
0.32	0.3995	0.1235	0.05144	0.4709
0.40	0.4037	0.1101	0.05630	0.4725
0.52	0.4096	0.08928	0.06419	0.4630
0.64	0.4156	0.06772	0.07288	0.4289
0.72	0.4199	0.05303	0.07907	0.3843
0.84	0.4269	0.03057	0.08888	0.2699
0.92	0.4321	0.01537	0.09571	0.1537
0.96	0.4348	0.00770	0.09919	0.08181
0.98	0.4362	0.00386	0.1010	0.04216
1.00	0.4376	0.0	0.1027	0.0

Table 6. "Converged" results for case 5.

μ	$I(0,-\mu)$	$-Q(0,-\mu)$
0.02	0.3633	0.1661
0.06	0.3739	0.1618
0.10	0.3810	0.1568
0.16	0.3886	0.1485
0.20	0.3924	0.1425
0.28	0.3983	0.1301
0.32	0.4008	0.1236
0.40	0.4052	0.1102
0.52	0.4112	0.08936
0.64	0.4175	0.06780
0.72	0.4220	0.05309
0.84	0.4294	0.03061
0.92	0.4348	0.01539
0.96	0.4377	0.007715
0.98	0.4392	0.003862
1.00	0.4407	0.0

yielded, not surprisingly, even better results for the integrated quantities A^* and B^*. We note that $\eta_0 \to \infty$ as $\omega \to 1$ and that the calculation of the quantities $\Gamma_\alpha(\eta_0)$ and $\Delta_\alpha(\eta_0)$ requires some care, for $\omega \simeq 1$, to avoid a loss of accuracy. Finally, we have observed that the method becomes less accurate for small thickness ($\tau_0 \simeq 0.1$); however, a modification in Eqs. (8) can improve the method for very thin media.

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