INSTITUTO DE ENERGIA ATOMICA - SÃO PAULO

AN APPROXIMATE SOLUTION OF THE TWO – GROUP CRITICAL PROBLEM FOR REFLECTED SLABS

Yuji Ishiguro and Roberto David M. Garcia

PUBL. IEA 478 CEN - AFR 49	MAIO/1977

PUBL. IEA 478
CEN - AFR 49

MAIO/1977

AN APPROXIMATE SOLUTION OF THE TWO-GROUP CRITICAL PROBLEM FOR REFLECTED SLABS

Yuji Ishiguro and Roberto David M. Garcia

CENTRO DE ENGENHARIA NUCLEAR
Área de Física de Reatores

INSTITUTO DE ENERGIA ATÔMICA
SÃO PAULO – BRASIL

APROVADO PARA PUBLICAÇÃO EM JANEIRO/1977

INSTITUTO DE ENERGIA ATÔMICA
Caixa Postal 11.049 (Pinheiros)
Cidade Universitária "Armando de Salles Oliveira"
SÃO PAULO — BRASIL

AN APPROXIMATE SOLUTION OF THE TWO-GROUP CRITICAL PROBLEM FOR REFLECTED SLABS

Yuji Ishiguro and Roberto David M. Garcia

ABSTRACT

A new approximation is developed to solve two group slab problems involving two media where one of the media is infinite. The method consists in combining the P_L approximation with invariance principles. Several numerical results are reported for the critical slab problem.

INTRODUCTION

A class of model problems in transport theory that have not been solved using the singular-eigenfunction expansion method is the so-called two-media problems in the two-group model. The main difficulty, as far as numerical calculations are concerned, is the fact that some of the eigenfunctions are singular and that these singularities cannot easily be removed.

Recently a method was proposed to reduce the case of two half-space media to regular computational form(4). However, for problems involving a finite medium, no methods have been found to regularize the ensuing equations.

Here we consider one of these problems, i.e., the critical problem for slab reactors with reflectors of infinite thickness. The problem is to find the critical core thickness while we assume the material properties are given. The solution in the core is approximated by the P_L solution. However, for the reflector, instead of utilizing the P_L solution, we resort to invariance principles and use the scattering function derived in exact transport theory(8).

ANALYSIS

The transport equation is of the form(4).

$$\mu \frac{\partial}{\partial x} \underset{\sim}{I}_i (x,\mu) + \underset{\sim}{\Sigma}_i \underset{\sim}{I}_i (x,\mu) = \underset{\sim}{Q}_i \int_{-1}^{1} \underset{\sim}{I}_i (x,\mu')d\mu' \tag{1}$$

where $i = 1$ refers to the core and $i = 2$ to the reflector and $\underset{\sim}{\Sigma}_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & 1 \end{bmatrix}$, $\sigma_i > 1$. We assume here that the matrix $\underset{\sim}{Q}_i$, with elements $q_{i\alpha\beta}$, is neither diagonal or triangular.

If we define 2 x 2 matrices $\underset{\sim}{P}_i$ by

$$\underset{\sim}{P}_i = \begin{bmatrix} \sqrt{q_{i21}} & 0 \\ 0 & \sqrt{q_{i12}} \end{bmatrix},$$

the solution of Eq (1) can be given by

$$\underset{\sim}{J}_1 = \underset{\sim}{P}_1^{-1}\, \underset{\sim}{\Psi}_1 (x, \mu)$$

where $\underset{\sim}{\Psi}_1 (x, \mu)$ is the solution of

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\Psi}_1 (x, \mu) + \underset{\sim}{\Sigma}_1 \underset{\sim}{\Psi}_1 (x, \mu) = \underset{\sim}{C}_1 \int_{-1}^{1} \underset{\sim}{\Psi}_1 (x, \mu')d\mu' \qquad (2)$$

with $\underset{\sim}{C}_1 = \underset{\sim}{P}_1 \underset{\sim}{Q}_1 \underset{\sim}{P}_1^{-1}$ being symmetric

The solution for the critical problem must satisfy the boundary conditions

$$\lim_{x \to \infty} \underset{\sim}{\Psi}_2 (x, \mu) = \underset{\sim}{0}\ ,$$

$$\underset{\sim}{\Psi}_1 (x, \mu) = \underset{\sim}{\Psi}_1 (-x, -\mu),\ \mu \in (-1,1),$$

and the continuity condition at the interface

$$\underset{\sim}{P}_1^{-1} \underset{\sim}{\Psi}_1 (a, \mu) = \underset{\sim}{E}\, \underset{\sim}{P}_2^{-1} \underset{\sim}{\Psi}_2 (a, \mu),\ \mu \in (0,1), \qquad (3a)$$

$$\underset{\sim}{P}_1^{-1} \underset{\sim}{\Psi}_1 (a, -\mu) = \underset{\sim}{E}\, \underset{\sim}{P}_2^{-1} \underset{\sim}{\Psi}_2 (a, -\mu),\ \mu \in (0,1), \qquad (3b)$$

where a is the required critical half-thickness of the core There are two cases for the $\underset{\sim}{E}$ matrix depending on the ordering of groups,

$$\underset{\sim}{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

In the following we consider only the first case, i.e., that the groups are similarly ordered.

Substituting the continuity condition given by Eqs (3) in the exact relation of invariance principles(8)

$$\underset{\sim}{\Psi}_2 (a, -\mu) = \frac{1}{2\mu} \int_0^1 \underset{\sim}{s}_2 (\mu, \mu')\, \underset{\sim}{\Psi}_2 (a, \mu')d\mu'$$

where $\underset{\sim}{s}_2$ is the scattering matrix for the reflector, we see that the core solution must satisfy the relation

$$\underset{\sim}{F}^{-1}\,\underset{\sim}{\Psi}_1(a,\mu)=\frac{1}{2\mu}\int_0^1 \underset{\sim}{s}_2(\mu,\mu')\,\underset{\sim}{F}^{-1}\,\underset{\sim}{\Psi}_1(a,\mu')d\mu',\ \mu\epsilon(0,1), \tag{4}$$

where $\underset{\sim}{F}=\underset{\sim}{P}_1\underset{\sim}{P}_2^{-1}$. Thus, to find the critical size, we only need to consider the core solution and the scattering function for the reflector. Subsequently we consider Eq. (4) as the basic relation to be satisfied.

If we introduce the transformation

$$\underset{\sim}{\Psi}_1(x,\mu)\rightarrow\begin{bmatrix}\theta(\mu) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Psi_{11}(x,\sigma_2\mu)\\ \Psi_{12}(x,\mu)\end{bmatrix}=\underset{\sim}{\theta}(\mu)\,\underset{\sim}{\Psi}_1^{*}(x,\mu),$$

where $\theta(\mu)=1$, $\mu\epsilon(-\frac{1}{\tau_2},\frac{1}{\tau_2})$, $\theta(\mu)=0$, otherwise, Eq. (4) becomes

$$\underset{\sim}{F}^{-1}\,\underset{\sim}{\theta}(\mu)\,\underset{\sim}{\Psi}_1^{*}(a,-\mu)=\underset{\sim}{\theta}(\mu)\,\underset{\sim}{\Sigma}_2^{-1}\frac{1}{2\mu}\int_0^1\underset{\sim}{S}_2(\mu,\mu')\underset{\sim}{F}^{-1}\,\underset{\sim}{\theta}(\mu')\,\underset{\sim}{\Sigma}_2\,\underset{\sim}{\Psi}_1^{*}(a,\mu')d\mu',\ \mu\epsilon(0,1), \tag{5}$$

with

$$\underset{\sim}{S}_2(\mu,\mu')=\frac{2\mu\mu'}{\mu+\mu'}\,\underset{\sim}{H}(\mu)\,\underset{\sim}{C}_2\,\underset{\sim}{\tilde{H}}(\mu').$$

Here $\underset{\sim}{H}(\mu)$ is the $\underset{\sim}{H}$ matrix for the reflector introduced by Siewert and Ishiguro[8] and a rapidly converging iterative scheme for its calculation is available. We can of course consider the exact solution $\underset{\sim}{\Psi}_1$ in Eq. (4), resulting in a set of singular integral equations. At present, however, no method is available to regularize them. We consider, therefore, an approximate solution for the core with an expectation that, since the reflector property is taken into account by the exact relation of invariance principles, the results will be more accurate than when the reflector solution is also approximated.

The symmetric P_L solution of Eq. (2) for the core can be written as

$$\underset{\sim}{\Psi}_1(x,\mu)={\sum_{j=1}^{L+1}}'A(\nu_j)\sum_{\ell=0}^{L}\frac{2\ell+1}{2\ell}P_\ell(\mu)\,\underset{\sim}{G}_\ell(\nu_j)\left[e^{-x/\nu_j}+(-1)^\ell e^{x/\nu_j}\right] \tag{6}$$

where $A(\nu_j)$ are expansion coefficients to be determined, $P_\ell(\mu)$ are Legendre polynomials, $\underset{\sim}{G}_\ell(\nu_j)$ are two-vector polynomials of ν_j (cf. Appendix), and ν_j are eigenvalues with ν_1 being imaginary and ν_j, $j=2,\ldots,L+1$, being real.

If we define

$$\dot{A}(\nu_1)=A(\nu_1),\ E_\ell(\nu_1,x)=e^{-x/\nu_1}+(-1)^\ell e^{x/\nu_1},$$

and for j > 1,

$$\hat{A}(\nu_j) = A(\nu_j)\, e^{a/\nu_j}\,,\quad E_\ell(\nu_j,x) = e^{-(x+a)/\nu_j} + (-1)^\ell e^{-(a-x)/\nu_j},$$

Eq. (6) becomes

$$\underset{\sim}{\Psi}_i(x,\mu) = \sum_{\ell=0}^{L} \frac{2\ell+1}{2} P_\ell(\mu)\, \underset{\sim}{K}_\ell(x,\hat{A})\,, \tag{7}$$

where

$$\underset{\sim}{K}_\ell(x,\hat{A}) = \sum_{j=1}^{L+1} \hat{A}(\nu_j)\, \underset{\sim}{G}_\ell(\nu_j)\, E_\ell(\nu_j,x) \tag{8}$$

Substituting Eq. (7) in Eq. (5) and using the relations

$$\int_0^1 \underset{\sim}{\tilde{H}}(\mu')\,\underset{\sim}{\theta}(\mu')\,\frac{\mu'}{\mu'+\mu}\,d\mu' = \underset{\sim}{C}_2^{-1}\,\underset{\sim}{H}^{-1}(\mu) + \underset{\sim}{\tilde{H}}_0 - \underset{\sim}{C}_2^{-1}$$

$$\int_0^1 \underset{\sim}{\tilde{H}}(\mu')\,\underset{\sim}{\theta}(\mu')\,\underset{\sim}{P}^*_\ell(\mu')\,\frac{\mu'}{\mu'+\mu}\,d\mu = [\,\underset{\sim}{C}^{-1}_2\,\underset{\sim}{H}^{-1}(\mu) + \underset{\sim}{\tilde{H}}_0 - \underset{\sim}{C}_2^{-1}\,]\,\underset{\sim}{P}^*_\ell(-\mu)$$

$$+ \sum_{i=1}^{\ell} a_{\ell,i} \sum_{k=1}^{i} (-1)^{k-1} \mu^{k-1}\, \underset{\sim}{\tilde{H}}_{i-k+1}\, \underset{\sim}{\Sigma}_2^{-1}\,,\quad \ell \geqslant 1\,,$$

where we have written

$$P_\ell(\mu) = \sum_{i=0}^{\ell} a_{\ell,i}\,\mu^i\,,$$

$$\underset{\sim}{H}_\alpha = \int_0^1 \underset{\sim}{\theta}(\mu)\,\underset{\sim}{H}(\mu)\,\mu^\alpha\,d\mu\,.$$

and

$$\underset{\sim}{P}^*_\ell(\mu) = \begin{bmatrix} P_\ell(\sigma_2\mu) & 0 \\ 0 & P_\ell(\mu) \end{bmatrix},$$

the boundary condition, Eq. (4), reduces to

$$\sum_{\ell=0}^{L} \frac{2\ell+1}{2}\, \underset{\sim}{R}_\ell(\mu)\, \underset{\sim}{F}^{-1}\, \underset{\sim}{K}_\ell(a,\hat{A}) = \underset{\sim}{0}\,,\ \mu \in (0,1)\,, \tag{9}$$

where

$$\underset{\sim}{R}_0(\mu) = \underset{\sim}{\theta}(\mu)\,\underset{\sim}{H}(\mu)\,[\underset{\sim}{C}_2\,\underset{\sim}{\tilde{H}}_0 - \underset{\sim}{I}\,]\,\underset{\sim}{\Sigma}_2$$

and, for $\ell \geqslant 1$,

$$\underset{\sim}{R}_\ell(\mu) = a_{\ell 0}\,\underset{\sim}{R}_0(\mu) + \sum_{i=1}^{\ell} a_{\ell i}\,\{(-1)^i\mu^i\,\underset{\sim}{R}_0(\mu)$$

$$+ \sum_{j=1}^{i}(-1)^{j-1}\mu^{i-1}\,\underset{\sim}{\theta}(\mu)\underset{\sim}{H}(\mu)\underset{\sim}{C}_2\underset{\sim}{\tilde{H}}_{i-j+1}\underset{\sim}{\Sigma}_2\,\}\underset{\sim}{\Sigma}_2^i\,.$$

Eq. (9), the boundary condition, cannot be satisfied as it stands for any finite L. Here we use the Marshak-type condition, i.e.,

$$\sum_{\ell=0}^{L}\frac{2\ell+1}{2}\int_0^1 \mu^k\underset{\sim}{R}_\ell(\mu)\,d\mu\;\underset{\sim}{F}^{-1}\,\underset{\sim}{K}_\ell(a,\hat{A}) = \underset{\sim}{0},\; k = 1,3,\ldots,L.$$

Thus, recalling Eq. (8), the expansion coefficients must satisfy

$$\sum_{j=1}^{L+1}\hat{A}(\nu_j)\sum_{\ell=0}^{L}\frac{2\ell+1}{2}\,\underset{\sim}{R}_{k\ell}\,\underset{\sim}{F}^{-1}\,\underset{\sim}{G}_\ell(\nu_j)\,\underset{\sim}{E}_\ell(\nu_j,a) = \underset{\sim}{0}\,, \tag{10}$$

where

$$\underset{\sim}{R}_{k\ell} = \int_0^1 \mu^k\,\underset{\sim}{R}_\ell(\mu)\,d\mu$$

can be written in terms of moments of the $\underset{\sim}{H}$ matrix as

$$\underset{\sim}{R}_{k0} = \underset{\sim}{H}_k\,[\underset{\sim}{C}_2\,\underset{\sim}{\tilde{H}}_0 - \underset{\sim}{I}\,]\,\underset{\sim}{\Sigma}_2$$

and, for $\ell \geqslant 1$,

$$\underset{\sim}{R}_{k\ell} = a_{\ell 0}\,\underset{\sim}{R}_{k0} + \sum_{i=1}^{\ell} a_{\ell i}\,\{(-1)^i\,\underset{\sim}{H}_{i+k}\,[\underset{\sim}{C}_2\,\underset{\sim}{\tilde{H}}_0 - \underset{\sim}{I}]$$

$$+ \sum_{j=1}^{i}(-1)^{j-1}\,\underset{\sim}{H}_{j+k-1}\,\underset{\sim}{C}_2\,\underset{\sim}{\tilde{H}}_{i-j+1}\,\}\underset{\sim}{\Sigma}_2^{i+1}\,.$$

Eq. (10) can be written in the form

$$\underset{\sim}{D}(a)\,\underset{\sim}{A} = \underset{\sim}{0}\,, \tag{11}$$

where $\underset{\sim}{A}$ is a L + 1 vector with components $\hat{A}(\nu_j)$ and $\underset{\sim}{D}(x)$ is a (L + 1) x (L + 1) matrix function whose components are given by

$$\begin{bmatrix} D_{kj} \\ D_{(k+1)j} \end{bmatrix} = \sum_{\ell=0}^{L} \frac{2\ell+1}{2} \underset{\sim}{R}_{k\ell} \underset{\sim}{F}^{-1} \underset{\sim}{G}_\ell(\nu_j) E_\ell(\nu_j,x) .$$

The critical half-thickness a can now be determined by the condition

$$\det \underset{\sim}{D}(a) = 0 . \qquad (12)$$

Finally, with an arbitrary normalization $\hat{A}(\nu_1) = 1$, the expansion coefficients are obtained by solving Eq. (11).

We note at this point that the critical condition, Eq. (12), is given as the determinant of a (L +1) x (L + 1) matrix, while in the usual method, where the solutions for both the core and reflector are approximated by P_L solutions, the corresponding matrix has the size (2L + 2) x (2L + 2).

NUMERICAL RESULTS

To report numerical comparisons of the two methods, we used seven sets of data for the core, listed in Tables I and III and four sets for the reflector, Tables II and IV and considered the eleven cases show in Table V for sample calculations. Some of the data sets are found in the literature, while sets C5, C6, C7 and S4 have been generated by us using the multigroup discrete-ordinate code XSDRN$^{(3)}$.

Table I

Two-Group Data Sets for Multiplying Media

Set ‡	Material	Energy Range		References
		Group 1	Group 2	
C1	not specified	not	specified	2,5
C2	not specified	not	specified	2,5
C3	Pu^{239}	0-1.35 MeV	1.35 MeV-∞	1, 2, 5
C4	U^{235}	0-1.35 MeV	1.35 MeV-∞	1, 2, 5
C5	$U^{235} + H_2O$; U/H = 1/300	0-0.3 eV	0.3 eV-15 MeV	
C6	$U^{235} + H_2O$; U/H = 1/500	0-0.3 eV	0.3 eV-15 MeV	
C7	$U^{235} + H_2O$; U/H = 1/1000	0-0.3 eV	0.3 eV-15 MeV	

The parameters $\underset{\sim}{\Sigma}$ and $\underset{\sim}{Q}$ in Eq.(1) are computed from the data sets by:

$$\sigma = \sigma_1 / \sigma_2$$

$$c_{ij} = \frac{1}{2\sigma_2} \{\sigma_{ijs} + \chi_i \nu_j \sigma_{jf}\}$$

Table II

Two-group Data Sets for Non-multiplying Media

Set	Material	Energy Range		References
		Group 1	Group 2	
S1	H_2O	not	specified	6
S2	Carbon	not	specified	6
S3	U^{238}	0-1.35 MeV	1.35 Mev- -	1, 2, 5
S4	H_2O	0-0.3 eV	0.3 eV-15 MeV	

Table III

Data Sets for the Core *

	C1	C2	C3	C4	C5	C6	C7
σ_1	0.54628	2.52025	0.3360	0.3456	2.9546	2.9628	2.9727
σ_2	0.33588	0.65696	0.2208	0.2160	0.88570	0.88655	0.88721
$\sigma_{11}{}^s$	0.42410	2.44383	0.23616	0.26304	2.8252	2.8751	2.9183
$\sigma_{12}{}^s$	0.0045552	0.029227	0.0432	0.0720	0.044115	0.045364	0.046352
$\sigma_{21}{}^s$	0.0	0.0	0.0	0.0	0.0016361	0.0011603	0.00076712
$\sigma_{22}{}^s$	0.31980	0.62568	0.0792	0.078240	0.83692	0.83807	0.83892
$\bar{\nu}_1\sigma_1{}^f$	0.2425	0.12658	0.2503392	0.17280	0.22992	0.14324	0.073910
$\bar{\nu}_2\sigma_2{}^f$	0.0070425	0.002621	0.29016	0.167184	0.0070013	0.0041243	0.0020883
χ_1	0.0	0.0	0.425	0.425	0.0	0.0	0.0
χ_2	1.0	1.0	0.575	0.575	1.0	1.0	1.0

* Cross sections in cm^{-1}

Table IV

Data Sets for Reflector *

	S1	S2	S3	S4
σ_1	0.595862	0.3552	0.3408	2.9865
σ_2	0.454908	0.28008	0.2208	0.88798
$\sigma_{11}{}^s$	0.522376	0.3520	0.33144	2.9676
$\sigma_{12}{}^s$	0.15364	0.0320	0.0984	0.047494
$\sigma_{21}{}^s$	0.054108	0.00296	0.0	0.00033616
$\sigma_{22}{}^s$	0.3006	0.2480	0.09552	0.83975
$\bar{\nu}_1\sigma_1{}^f$	–	–	0.0005928	–
$\bar{\nu}_2\sigma_2{}^f$	–	–	0.0653952	–
χ_1	–	–	0.425	–
χ_2	–	–	0.575	–

* *Cross sections in* cm^{-1}

For each case the critical half thickness was calculated in the usual P_L method ($P_L - P_L$) and in our method ($P_L - H$).

The results are listed in Table VI, together with some "exact" values obtained by numerical methods[(6)] (e.g., high-order S_N method) as a basis of comparison. Normalized scalar fluxes in the core obtained in P_1 - H and P_7 - H approximations are show in Figures 1, 2 and 3 for three cases:

Figure 1 Small fast reactor

Figure 2 Small thermal reactor

Figure 3 Large thermall reactor

Table V

Cases Considered

Case	Core	Reflector
1	C1	S1
2	C1	S2
3	C2	S1
4	C2	S2
5	C3	S1
6	C3	S2
7	C3	S3
8	C4	S3
9	C5	S4
10	C6	S4
11	C7	S4

In many cases the difference in the scalar flux between the two methods is quite small and for large reactors, e.g., Case 11, P_1 solutions are as accurate as P_7 results. However, for small reactors, e.g. Case 8, for which diffusion theory is known to be inaccurate low order approximations are poor and the P_L - H approximation is better than the usual P_L method, as shown in Figure 4.

COMMENTS AND CONCLUSIONS

Our aim in this work was to develop a new approximate method in which better results could be obtained with equivalent or less amount of computational work as compared to the usual P_L method. The basis of this expectation is that, to find the critical size, we only need to consider L + 1 unknowns, $A(\nu_j)$, and thus the critical matrix is of half the size and that in our method only the core solution is approximated.

The results of sample calculations, however, are not very encouraging, though in most cases our results are better than those by the usual P_L method. One possible explanation is that, although the effect of the reflector is taken into account using the result of exact transport theory, the approximate nature of the core solution is faithfully reflected the use of the scattering matrix thus that the reflector solution is, in effect, approximated. Of course, if this is the reason, it should also apply to one-group theory; we do not really know why in the one-group model the P_L - H approximation is consistently better[(7)] but not in two-group theory.

A merit of our method can still be found in the fact that the critical matrix is smaller when used in a parametric survey of the critical size where the reflector material is fixed and the core parameters are changed, since the matrix $\underset{\sim}{B}_{k\ell}$ in Eq (10) are independent of the core property and thus can be calculated once and for all.

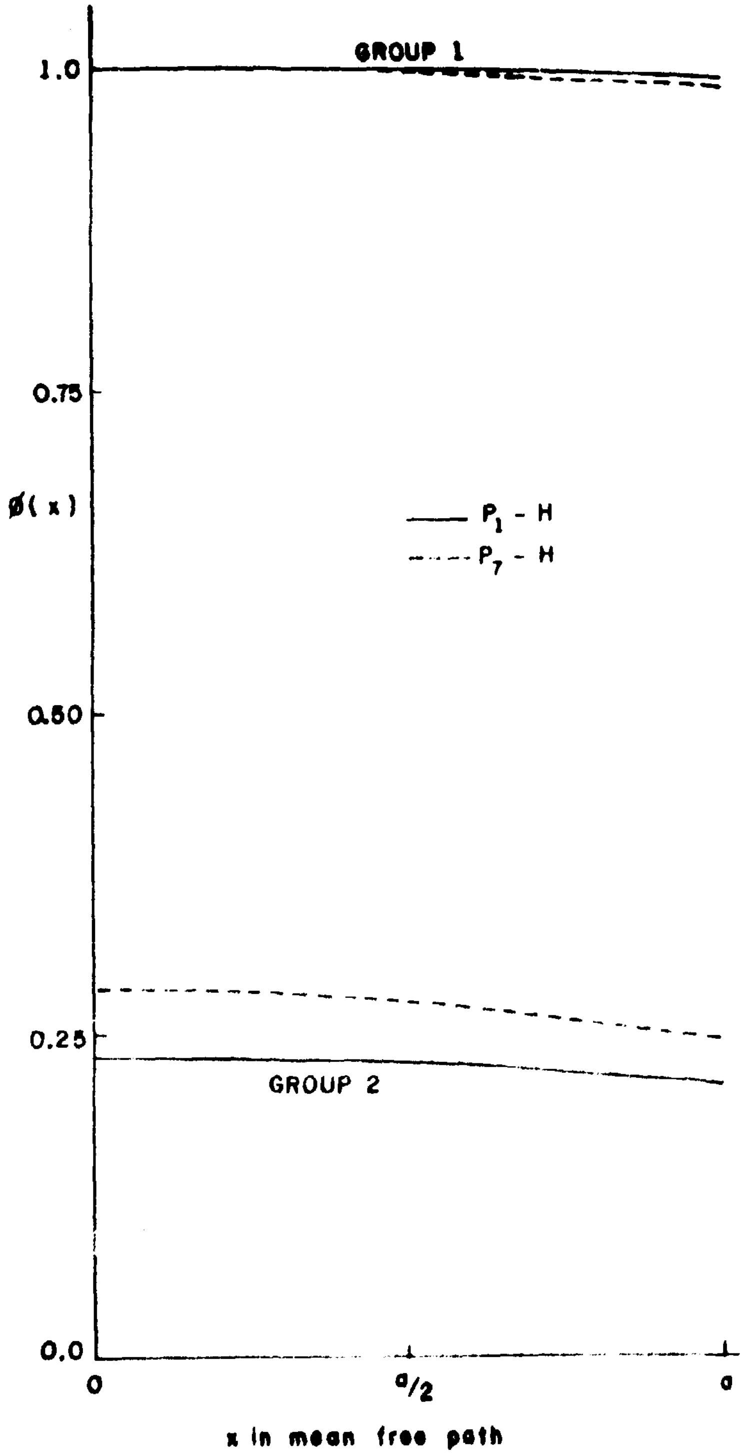

Figure 1 – Scalar fulxes in the core – Case 8

Table VI

Critical Half-thickness *

Case #	L = 1	L = 3	L = 5	L = 7	Exact
1	240.67	258.03	259.81	260.35	260.740
	257.33	259.98	260.53	260.72	
2	0.054277	0.055049	0.055517	0.055805	0.0564400
	0.054620	0.055339	0.055777	0.056010	
3	3.1233	3.0713	3.0714	3.0716	3.07186
	3.1225	3.0730	3.0721	3.0720	
4	0.26582	0.26769	0.26803	0.26814	0.268285
	0.26767	0.26806	0.26817	0.26822	
5	0.12175	0.11025	0.10701	0.10543	0.102562
	0.12105	0.10878	0.10572	0.10440	
6	0.020717	0.020233	0.020068	0.019968	0.0195199
	0.020717	0.020158	0.019978	0.019875	
7	0.10395	0.095084	0.092659	0.091451	
	0.10426	0.094208	0.091746	0.090673	
8	0.17181	0.15941	0.15670	0.15554	
	0.17281	0.15867	0.15596	0.15498	
9	1.5594	1.5194	1.5188	1.5187	
	1.5625	1.5200	1.5190	1.5188	
10	2.2243	2.1825	2.1821	2.1821	
	2.2244	2.1831	2.1823	2.1822	
11	4.1992	4.1576	4.1574	4.1574	
	4.1940	4.1583	4.1576	4.1575	

* given in units of m.f.p. of group 2 in the core.

** For each case, upper figure refers to $P_L - P_L$ result and lower figure to $P_L - H$

APPENDIX

The P_L solution of the two-group transport equation

The P_L solution of Eq. (2) is written as

$$\underset{\sim}{\Psi}(x,\mu) = \sum_{\ell=0}^{L} \frac{2\ell+1}{2} P_\ell(\mu)\, \underset{\sim}{K_\ell}(x) \qquad \text{(A-1)}$$

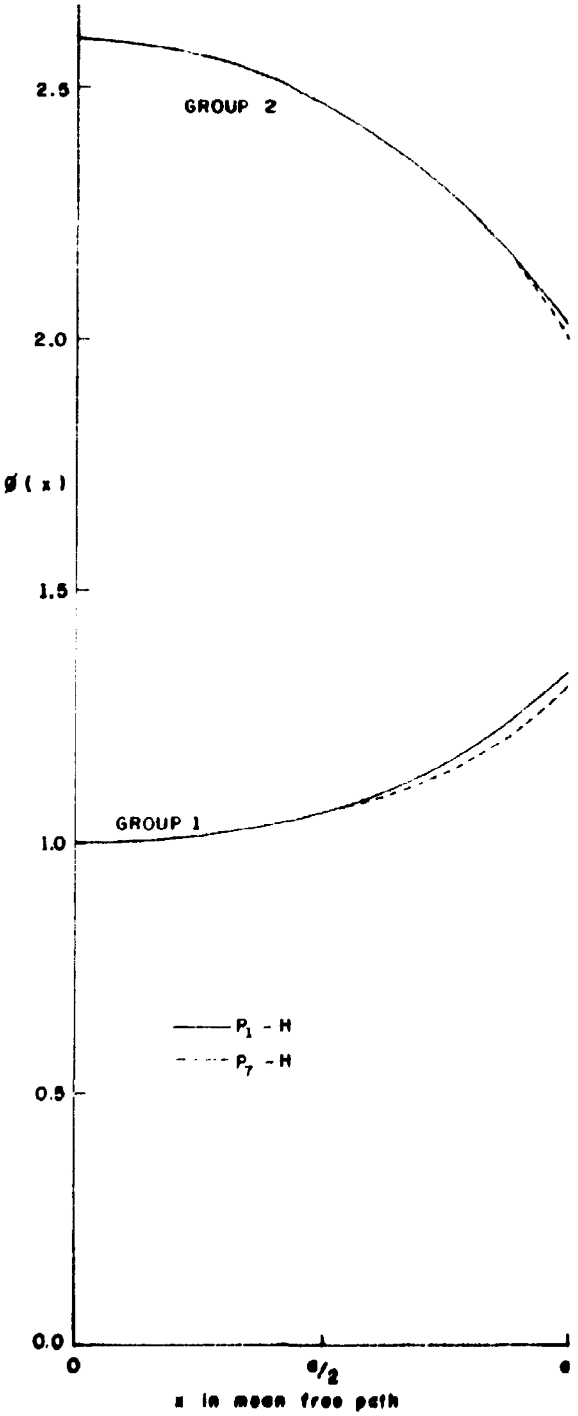

Figure 2 – Scalar Fluxes in the core – Case 9

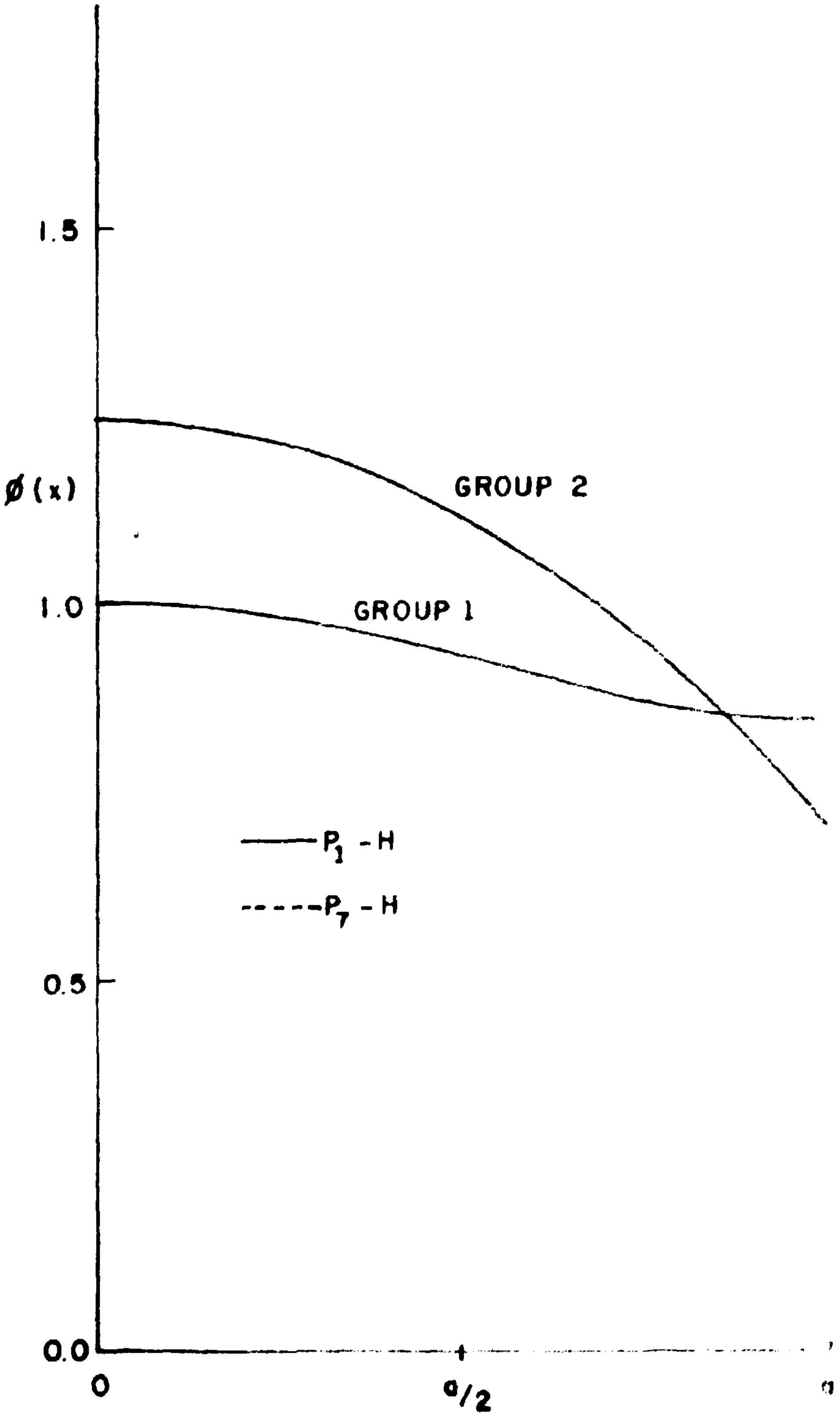

Figure 3 – Scalar fluxes in the core – Case 11

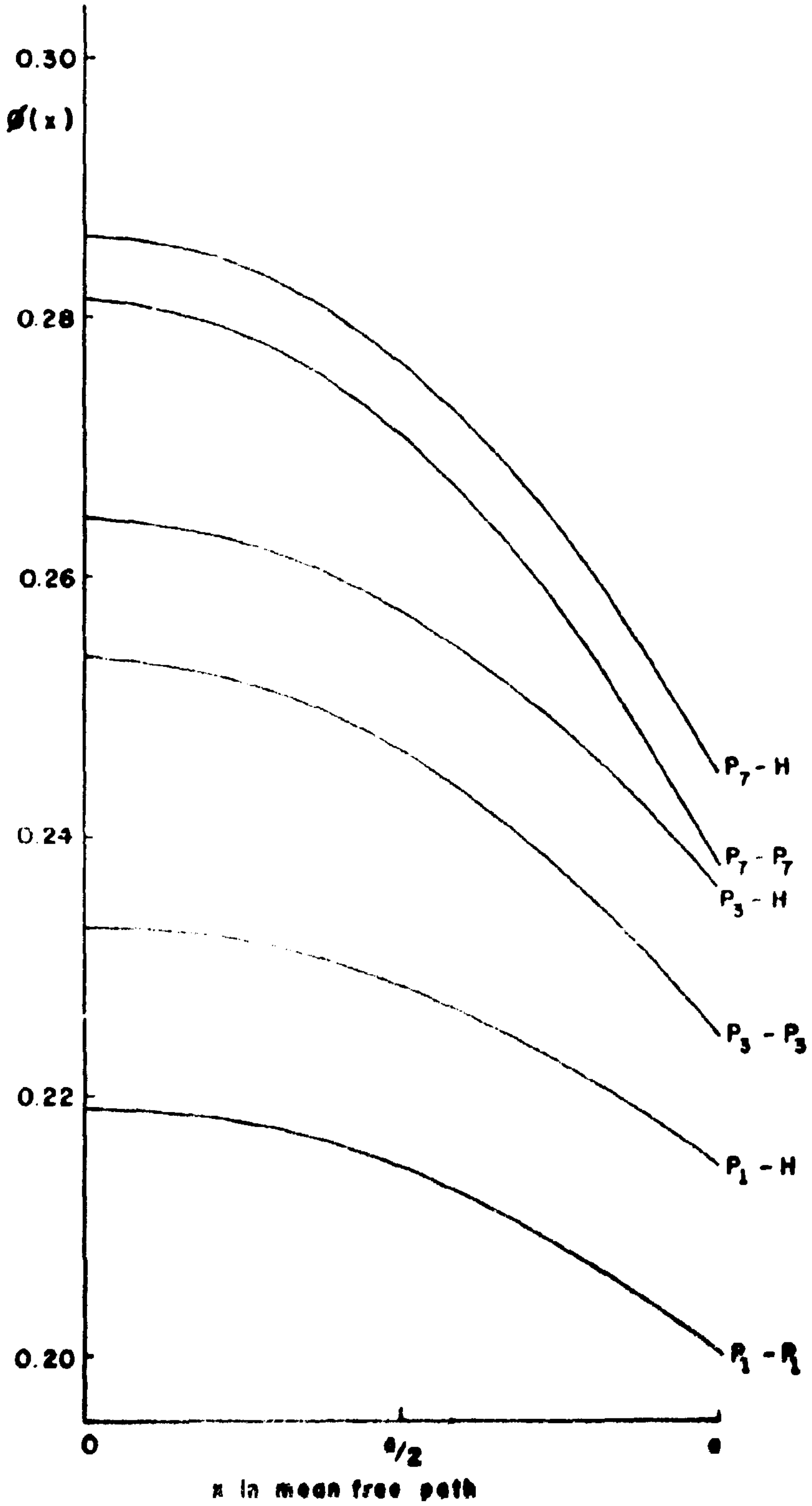

Figure 4 — Group 2 scalar flux — Case 8

where we consider L to be odd.

Substituting the above in Eq. (2) and using the recurrence relation of Legendre polynomials, we obtain

$$\sum_{\ell=0}^{L} \left[(\ell+1) P_{\ell+1}(\mu) + \ell P_{\ell-1}(\mu) \right] \underset{\sim}{K}'_{\ell}(x)$$

$$+ \sum_{\ell=0}^{L} (2\ell+1) P_{\ell}(\mu)\, \underset{\sim}{\Sigma}\, \underset{\sim}{K}_{\ell}(x) = 2\, \underset{\sim}{C}\, \underset{\sim}{K}_{o}(x). \qquad \text{(A-2)}$$

If we multiply Eq. (A-2) by $P_n(\mu)$ and integrate over $\mu \epsilon (-1,1)$, we obtain

$$n = 0 \qquad \underset{\sim}{K}'_{1}(x) + [\underset{\sim}{\Sigma} - 2\underset{\sim}{C}]\, \underset{\sim}{K}_{o}(x) = \underset{\sim}{0} \qquad \text{(A-3a)}$$

$$1 \leqslant n \leqslant L-1 \qquad \underset{\sim}{K}'_{n+1}(x) + \frac{2n+1}{n+1}\, \underset{\sim}{\Sigma}\, \underset{\sim}{K}_{n}(x) + \frac{n}{n+1}\, \underset{\sim}{K}'_{n-1}(x) = \underset{\sim}{0} \qquad \text{(A-3b)}$$

$$n = L \qquad (2L+1)\, \underset{\sim}{\Sigma}\, \underset{\sim}{K}_{L}(x) + L\, \underset{\sim}{K}'_{L-1}(x) = \underset{\sim}{0} \qquad \text{(A-3c)}$$

where we have taken $\underset{\sim}{K}_{L+1}(x) = \underset{\sim}{0}$.

If we write

$$\underset{\sim}{K}_{\ell}(x) = \underset{\sim}{G}_{\ell}(\nu)\, e^{-x/\nu}$$

Eqs. (A-3) reduce to

$$-\frac{1}{\nu}\, \underset{\sim}{G}_{1}(\nu) + [\underset{\sim}{\Sigma} - 2C]\, \underset{\sim}{G}_{o}(\nu) = \underset{\sim}{0} \qquad \text{(A-4a)}$$

$$-\frac{1}{\nu}\, \underset{\sim}{G}_{n+1}(\nu) + \frac{2n+1}{n+1}\, \underset{\sim}{\Sigma}\, \underset{\sim}{G}_{n}(\nu) - \frac{1}{\nu}\, \frac{n}{n+1}\, \underset{\sim}{G}_{n-1}(\nu) = \underset{\sim}{0} \qquad \text{(A-4b)}$$

$$(2L+1)\, \underset{\sim}{\Sigma}\, \underset{\sim}{G}_{L}(\nu) - \frac{1}{\nu}\, L\, \underset{\sim}{G}_{L-1}(\nu) = \underset{\sim}{0} \qquad \text{(A-4c)}$$

Since the set of Eqs. (4) is homogeneous, we can write

$$\underset{\sim}{G}_{n}(\nu) = \underset{\sim}{M}_{n}(\nu)\, \underset{\sim}{G}_{o}(\nu), \quad \underset{\sim}{M}_{o}(\nu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{(A-5)}$$

and obtain

$$\left[-\frac{1}{\nu} \utilde{M}_1(\nu) + \utilde{\Sigma} - 2\utilde{C} \right] \utilde{G}_o(\nu) = \utilde{0} \qquad \text{(A-6a)}$$

$$\left[-\frac{1}{\nu} \utilde{M}_{n+1}(\nu) + \frac{2n+1}{n+1} \utilde{\Sigma} \utilde{M}_n(\nu) - \frac{1}{\nu} \frac{n}{n+1} \utilde{M}_{n-1}(\nu) \right] \utilde{G}_o(\nu) = \utilde{0} \qquad \text{(A-6b)}$$

$$\left[(2L+1) \utilde{\Sigma} \utilde{M}_L(\nu) - \frac{1}{\nu} L \utilde{M}_{L-1}(\nu) \right] \utilde{G}_o(\nu) = \utilde{0} \qquad \text{(A-6c)}$$

Thus the eigenvalues are determined by

$$\det\left[(2L+1) \utilde{\Sigma} \utilde{M}_L(\nu) - \frac{1}{\nu} L \utilde{M}_{L-1}(\nu) \right] = 0 \qquad \text{(A-7)}$$

or, formally,

$$\det \utilde{M}_{L+1}(\nu) = 0 .$$

Eq. (7) is a (L + 1)th order polynomial of ν^2 and thus the eigenvalues appear in L + 1 pairs $\pm \nu_1, \pm \nu_2, \ldots, \pm \nu_{L+1}$. It can also be shown that when $\det[\,\utilde{\Sigma} - 2\utilde{C}\,] < 0$, at least one eigenvalue is imaginary. For each eigenvalue pair, $\pm \nu_j$, we can determine, by Eqs. (A-6), corresponding $\utilde{G}_o(\nu_j)$ and $\utilde{M}_\ell(\pm \nu_j)$, and thus the vectors $\utilde{G}_\ell(\pm \nu_j)$ by Eq. (A-5).

However, if any of the eigenvalues is such that $|\nu_j| >> 1$, the above determination of $\utilde{G}_\ell(\pm \nu_j)$ is not satisfactory, because the recurrence relation for $\utilde{M}_\ell$ is such that the absolute values of its elements are increasing with ℓ. It can easily be seen that the elements of vectors $\utilde{G}_\ell$ must be decreasing, in absolute value, with increasing ℓ. This holds true in the above derivation of $\utilde{G}_\ell$. The problem is that in Eq. (A-5) elements of $\utilde{G}_n$ are given by the difference of two large numbers and thus that the accuracy of computation is lost. To avoid this loss of accuracy, we can derive a set of $\utilde{G}_\ell$ that is decreasing with ℓ.

In Eq. (A-4c) we define

$$\utilde{\mathcal{G}}_L(\nu) = \frac{1}{\nu} \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$$

and thus obtain

$$\utilde{\mathcal{G}}_{L-1}(\nu) = \frac{2L+1}{L} \begin{bmatrix} \sigma \\ 1 \end{bmatrix}$$

Eq. (A-4b) can now be used to derive all other $\utilde{G}_\ell$;

$$\underset{\sim}{G}_{n-1}(\nu) = \frac{2n+1}{n}\,\nu\,\underset{\sim}{\Sigma}\,\underset{\sim}{G}_n(\nu) - \frac{n+1}{n}\,\underset{\sim}{G}_{n+1}(\nu),\ n = L-1,\ldots,1.$$

Eq. (A-4a) is satisfied identically for all eigenvalues determined by Eq. (A-7). Finally, the vectors $\underset{\sim}{G}_\ell$ derived above can be normalized by dividing all elements by, for example, the group 1 element of $\underset{\sim}{G}_o$.

With the vectors $\underset{\sim}{G}_\ell$ thus determined, we can write

$$\underset{\sim}{K}_\ell(x) = \sum_{j=1}^{L+1} \left[A(\nu_j)\,\underset{\sim}{G}_\ell(\nu_j)\,e^{-x/\nu_j} + A(-\nu_j)\,\underset{\sim}{G}_\ell(-\nu_j)\,e^{x/\nu_j} \right]$$

and the general P_L solution of the transport equation is given by Eq. (A-1).

ACKNOWLEDGEMENT

The authors are grateful to Dr. K. D. Lathrop at Los Alamos Scientific Laboratory for supplying some of the data sets for reflector and also for obtaining exact values for some of the cases and to Dr. C. E. Siewert of North Carolina State University for initiating this problem and for various discussions during the course of this work.

RESUMO

Uma nova aproximação é desenvolvida para resolver o problema da placa em dois grupos envolvendo dois meios, um dos quais é infinito. O método consiste numa combinação da aproximação P_L com princípios de invariança. São apresentados vários resultados numéricos para o problema da placa crítica.

RÉSUMÉ

On a développé une nouvelle aproximation pour résoudre le problème d'une plaque, à deux groupes, avec deux matériaux, l'un d'extension infinie. La méthode consiste dans une combinaison de l'aproximation P_L avec les principes d'invariance. On a apresenté des résultats numériques pour le problème de la plaque critique.

REFERENCES

1. ARGONNE NATIONAL LABORATORY, Argonne, Ill. *Reactor physics constants*. 2.ed. Argonne, Ill., July 1963. (ANL-5800). p.581.

2. FORSTER, R. A. & METCALF, D. R. Two-group transport solutions to the one-dimensional critical slab problem. *Trans. Am. nucl. Soc.*, Hinsdale, Ill., 12:637-8, 1969.

3. GREENE, N. M. & CRAVEN, C. N. *XSDRN: a discrete ordinates spectral averaging códe*. Oak Ridge, Oak Ridge National Laboratory, July 1969. (ORNL-TM-2500).

4. ISHIGURO, Y. & MAIORINO, J. R. Two half-space Milne problem in two-group neutron-transport theory. *Trans. Am. nucl. Soc.*, Hinsdale, Ill., 22:253-4, 1975.

5. KRIESE, J. T.; SIEWERT, C. E. & YENER, Y. Two-group critical problem for slabs and spheres in neutron-transport theory. *Nucl. Sci. Engng.*, New York, 50:3-9, 1973.

6. LATHROP, K. D. Private communication.

7. NESHAT, K.; SIEWERT, C. E. & ISHIGURO, Y. An improved P_L solution to the reflected critical reactor problem in slab geometry. *Nucl. Sci. Engng.*, in print.

8. SIEWERT, C. E. & ISHIGURO, Y. Two-group neutron-transport theory: half-range orthogonality, normalization integrals, applications and computations. *J. nucl. Energy*, London, 26:251-69, 1972.