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The differential perturbative method applied to the sensitivity analysis for waterhammer problems in hydraulic networks

Jorge L. Baliño [a,*], Axel E. Larreteguy [a,1], Alicia C. Lorenzo [b], Alberto Gallardo Padilla [c], Fernando R. de Andrade Lima [d]

[a] *Centro Atómico Bariloche (Comisión Nacional de Energía Atómica), e Instituto Balseiro (Universidad Nacional de Cuyo), S.C. de Bariloche, Argentina*
[b] *Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil*
[c] *Instituto Peruano de Energía Nuclear, Lima, Peru*
[d] *Universidade Federal de Pernambuco, Recife, Brazil*

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Abstract

The differential perturbative method was applied to the sensitivity analysis for waterhammer problems in hydraulic networks. Starting from the classical waterhammer equations in a single-phase liquid with friction (the direct problem) the state vector comprising the piezometric head and the velocity was defined. Applying the differential method the adjoint operator, the adjoint equations with the general form of their boundary conditions, and the general form of the bilinear concomitant were calculated for a single pipe. Considering that any hydraulic network can be built by connecting different components (reservoirs, valves, pumps, tees, etc.) through pipes, the adjoint relationships for any component, as well as the final contribution to the bilinear concomitant, were calculated. Moreover, an analogy was established in which transmission and reflection coefficients can be derived for any adjoint component. The importance or adjoint function was analyzed when the piezometric head or velocity at a given position and time is chosen as the response functional. In this case, it is shown that the importance function is represented by delta-functions travelling along the hydraulic network with the propagation speed. The calculation of the sensitivity coefficients takes into account the cases in which the parameters under consideration influence the initial condition. For these cases, the calculation can be performed by solving sequentially two perturbative problems: the first one is non-steady, while the second one is steady, with an appropriate selection of a weight function coming from the unsteady perturbative problem. The discretized adjoint equations and the corresponding boundary conditions were programmed and solved by using the method of characteristics. As an example, a constant-level tank connected through a pipe to a valve discharging to atmosphere was considered. The corresponding sensitivity coefficients due to the variation of different parameters by using both the differential method and the response surface generated by the computer code WHAT, solver of the direct problem, were also calculated. The results obtained with these methods show excellent agreement. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Differential perturbative method; Sensitivity analysis; Waterhammer; Hydraulic networks

* Corresponding author. Currently at Instituto de Pesquisas Energéticas e Nucleares (IPEN), São Paulo, Brazil.
E-mail addresses: jlbalino@net.ipen.br (J.L. Baliño), alarreteguy@uade.edu.ar (A.E. Larreteguy), alicia@thor.biof.ufrj.br (A.C. Lorenzo), gallardo@ipencn.gov.pe (A.G. Padilla), f_andradelima@uol.com.br (F.R. de Andrade Lima).
[1] Currently at Universidad Argentina de la Empresa, Buenos Aires, Argentina.

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1. Introduction

The analysis of waterhammer transients plays an essential role in diverse areas such as hydroelectric projects, pumped-storage schemes, water supply systems, nuclear power plants, oil pipelines and industrial piping systems [1]. In nuclear power plants, for example, various operating transients in the heat transport system can lead to significant pressure changes, which must be taken into account in the design for a safe operation [2].

Coupled to any transient calculation, there is a necessity of estimating the influence of changes in the parameters on a defined response based on the obtained solution. This task, known as sensitivity analysis, can be performed by running repeatedly the computer code used in the calculations for different values of the parameters; in this way, a response surface is generated. However, this method requires time consuming calculations when many parameters are involved.

A different approach to perform the sensitivity analyses are the perturbation methods, which have been extensively used in reactor physics through the concept of the importance function [3–5]. The application of perturbation methods to the thermal-hydraulics field has been first proposed by Oblow [6] by using the so-called differential method; since then, it has been successfully extended [7]. The differential method has the following advantages:

(i) The sensitivity analysis can be performed without choosing a priori any parameter.

(ii) The calculations are faster and more efficient, since only one additional set of linear equations needs to be solved for a prescribed response.

The differential method is restricted to the linear behavior of the response surface in the vicinity of a specific design point, this being the main disadvantage.

The purpose of this paper is to outline the development of the sensitivity theory for a general waterhammer problem. The organization of this paper proceeds as follows: in Section 2 the theory for a general waterhammer problem is presented. In Section 3 the boundary conditions for the adjoint problem and the bilinear concomitant are derived, taking into account that a hydraulic network can be built with pipes connecting components. In Section 4 some response functionals are analyzed, namely the local instantaneous piezometric head and velocity, and the adjoint reflection and transmission coefficients are calculated for different components. In Section 5 the numerical procedure (i.e., the method of characteristics) used to solve the direct and adjoint problems, as well as the numerical difficulties, is outlined. In Section 6 a simple example is presented, showing an application of the methodology.

2. Theory for a general waterhammer problem

2.1. Direct equations

Consider the general one-dimensional waterhammer equations in a single-phase liquid with friction [8], in which the convective terms are neglected:

$$m_1 \equiv \frac{\partial H}{\partial t} + V \sin\theta + \frac{a^2}{g}\frac{\partial V}{\partial x} = 0, \tag{1}$$

$$m_2 \equiv g\frac{\partial H}{\partial x} + \frac{\partial V}{\partial t} + \zeta\frac{V|V|}{2D} = 0, \tag{2}$$

where H is the piezometric head, V is the fluid velocity, a is the wave propagation speed, D is the pipe diameter, θ is the pipe inclination angle, g is the gravity acceleration and ζ is the Darcy friction factor (function of V).

The piezometric head is defined as

$$H = \frac{p}{\rho g} + z, \tag{3}$$

where p is the pressure, ρ is the fluid density and z is the level.

The propagation speed a depends on the elastic properties of the fluid and pipe material, as well as on geometry. It can be calculated as

$$a = \left[\rho\left(\frac{1}{K} + \frac{C_1 D}{eE}\right)\right]^{-1/2}, \tag{4}$$

where K is the elasticity modulus, e is the pipe thickness and E is Young's modulus. C_1 is a non-dimensional coefficient related to the way the pipe undergoes elastic deformation:

$C_1 = (5/4) - \mu$ for a pipe anchored at the upper end, free to move in a longitudinal direction throughout its length, and without expansion joints.

$C_1 = 1 - \mu^2$ for a pipe which is anchored against longitudinal movement throughout its length.

$C_1 = 1 - (\mu/2)$ for a pipe which has expansion joints between anchors throughout the length of the pipe, where μ is Poisson's ratio for the pipe wall material.

The Darcy friction factor ζ can be calculated as [16]:

$$\zeta = \frac{64}{Re_d} \quad \text{for } Re_d < 2000 \ \text{(laminar flow)}, \tag{5}$$

$$\zeta = \left[-1.8\log_{10}\left(\frac{6.9}{Re_d} + \left(\frac{\varepsilon/D}{3.7}\right)^{1.11}\right)\right]^{-2} \quad \text{for } Re_d > 2000 \ \text{(turbulent flow)}. \tag{6}$$

In Eq. (6) ε is the pipe roughness and Re_d is the Reynolds number, defined as

$$Re_d = \frac{|V|D}{\nu}, \tag{7}$$

where ν is the kinematic viscosity.

The waterhammer equations can be written in a general way as

$$\boldsymbol{m}(\boldsymbol{f}, \boldsymbol{p}) = \boldsymbol{0}, \tag{8}$$

where $\boldsymbol{m}$ is a nonlinear operator, $\boldsymbol{f} = [H, V]^{\mathrm{T}}$ is the state vector and $\boldsymbol{p} = [p_1, p_2, \ldots, p_I]^{\mathrm{T}}$ is the parameter vector, which in turn depends on the generalized coordinate vector $\boldsymbol{r} = [x, t]^{\mathrm{T}}$. The parameter vector can represent physical constants and other parameters related to the initial or boundary conditions.

The corresponding boundary and initial conditions at the domain surface $\boldsymbol{r}_s$ can be written as

$$\boldsymbol{C}(\boldsymbol{f}, \boldsymbol{p}) = \boldsymbol{0} \quad \text{at } \boldsymbol{r} = \boldsymbol{r}_s. \tag{9}$$

Consider now a response functional R given by the expression:

$$R = \langle \boldsymbol{S}^+ \cdot \boldsymbol{f} \rangle, \tag{10}$$

where $\boldsymbol{S}^+ \equiv [S_H^+, S_V^+]^{\mathrm{T}}$ is an assigned vector weight function, while the brackets represent integration over the whole domain Ω. By now, this domain can be thought as related to a single pipe of length X; if the observation time is T, then $\Omega = [0, X] \times [0, T]$. In Section 3 we shall extend the concept to a general hydraulic network. In the following, we shall look for an expression for the change δR in the response functional in terms of the perturbations δp_i of the system parameters. In particular, expressions giving the sensitivity coefficients relevant to different parameters will be obtained.

2.2. Derived equations

Expanding Eq. (8) around a reference solution and considering the perturbations δp_i as independent, we obtain the general expression for the derived equations:

$$\underline{\boldsymbol{H}}\boldsymbol{f}_{/i} = \boldsymbol{S}(i), \tag{11}$$

where

$$\boldsymbol{f}_{/i} = \frac{\partial \boldsymbol{f}}{\partial p_i} = \left[H_{/i}, V_{/i}\right]^{\mathrm{T}}, \tag{12}$$

$$\boldsymbol{S}(i) = -\frac{\partial \boldsymbol{m}}{\partial p_i} = [S_H(i), S_V(i)]^{\mathrm{T}}, \tag{13}$$

$$\underline{\boldsymbol{H}} = \frac{\overline{\partial}\boldsymbol{m}}{\partial \boldsymbol{f}}. \tag{14}$$

The symbol $\overline{\partial}$ denotes a Frechet derivative [9]. From Eqs. (1), (2) and (14), we get for the forward operator $\underline{\boldsymbol{H}}$:

$$\underline{\boldsymbol{H}} = \begin{bmatrix} \dfrac{\partial(\cdot)}{\partial t} & (\cdot)\sin\theta + \dfrac{a^2}{g}\dfrac{\partial(\cdot)}{\partial x} \\ g\dfrac{\partial(\cdot)}{\partial x} & \dfrac{\partial(\cdot)}{\partial t} + (\cdot)\dfrac{|V|}{D}\left(\zeta + \dfrac{V}{2}\dfrac{\partial\zeta}{\partial V}\right) \end{bmatrix}. \tag{15}$$

The final derived waterhammer equations are then:

$$\frac{\partial H_{/i}}{\partial t} + V_{/i}\sin\theta + \frac{a^2}{g}\frac{\partial V_{/i}}{\partial x} = S_H(i), \tag{16}$$

$$g\frac{\partial H_{/i}}{\partial x} + \frac{\partial V_{/i}}{\partial t} + V_{/i}\frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial\zeta}{\partial V}\right) = S_V(i). \tag{17}$$

The corresponding boundary conditions for Eqs. (16) and (17) can be derived from Eq. (9) as:

$$\boldsymbol{C}_{/i} + \frac{\overline{\partial}\boldsymbol{C}}{\partial \boldsymbol{f}}\boldsymbol{f}_{/i} = \boldsymbol{0} \quad \text{at } \boldsymbol{r} = \boldsymbol{r}_s, \tag{18}$$

where

$$\boldsymbol{C}_{/i} = \frac{\partial \boldsymbol{C}}{\partial p_i}. \tag{19}$$

Expanding Eq. (10) to a first order, we obtain the change in the response functional as

$$\delta R = \sum_{i=1}^{I} \delta p_i\Big(\langle \boldsymbol{S}^{+}_{/i}\cdot\boldsymbol{f}\rangle + \langle \boldsymbol{S}^{+}\cdot\boldsymbol{f}_{/i}\rangle\Big). \tag{20}$$

2.3. Adjoint equations

According to Eq. (20), it would be possible in principle to obtain the change in the response functional by solving for each parameter the corresponding linear system of derived equations. Instead, the concept of the adjoint function [10] can be used, defined by

$$\langle \boldsymbol{f}_{/i} \cdot (\underline{\boldsymbol{H}}^{*}\boldsymbol{f}^{*})\rangle = \langle \boldsymbol{f}^{*} \cdot (\underline{\boldsymbol{H}}\boldsymbol{f}_{/i})\rangle + P(\boldsymbol{f}^{*},\boldsymbol{f}_{/i}), \tag{21}$$

where $\boldsymbol{f}^{*}$ is the adjoint or importance function and $\underline{\boldsymbol{H}}^{*}$ is the corresponding adjoint operator. The bilinear concomitant P accounts for the occurrence of non-homogeneous boundary conditions in Eq. (4). The linear adjoint system is required to satisfy

$$\underline{\boldsymbol{H}}^{*}\boldsymbol{f}^{*} = \boldsymbol{S}^{+} \tag{22}$$

with boundary condition

$$\boldsymbol{C}^{*}(\boldsymbol{f}^{*},\boldsymbol{p}) = \boldsymbol{0} \quad \text{at } \boldsymbol{r} = \boldsymbol{r}_{s}. \tag{23}$$

The particular form of the bilinear concomitant and the adjoint boundary conditions will be outlined in Section 3.

According to Eqs. (11), (20)–(22) the change in the response functional can now be obtained as

$$\delta R = \sum_{i=1}^{I} \delta p_i \Big[\langle \boldsymbol{S}^{+}_{/i} \cdot \boldsymbol{f} \rangle + \langle \boldsymbol{f}^{*} \cdot \boldsymbol{S}(i) \rangle + P \Big]. \tag{24}$$

Applying the definition of Eq. (21) and performing integration by parts when needed (see [6]), we get

$$\underline{\boldsymbol{H}}^{*} = \begin{bmatrix} -\dfrac{\partial(\cdot)}{\partial t} & -g\dfrac{\partial(\cdot)}{\partial x} \\ -\dfrac{a^2}{g}\dfrac{\partial(\cdot)}{\partial x} + (\cdot)\sin\theta & -\dfrac{\partial(\cdot)}{\partial t} + (\cdot)\dfrac{|V|}{D}\left(\zeta + \dfrac{V}{2}\dfrac{\partial \zeta}{\partial V}\right) \end{bmatrix}, \tag{25}$$

$$P = -\int_0^X \left(H^{*}H_{/i} + V^{*}V_{/i}\right)\Big|_0^T \mathrm{d}x - \int_0^T \left[H_{/i} g V^{*} + V_{/i}\frac{a^2}{g}H^{*}\right]\Bigg|_0^X \mathrm{d}t. \tag{26}$$

For the adjoint equations, we finally get:

$$-\frac{\partial H^{*}}{\partial t} - g\frac{\partial V^{*}}{\partial x} = S^{+}_{H}, \tag{27}$$

$$-\frac{a^2}{g}\frac{\partial H^{*}}{\partial x} + H^{*}\sin\theta - \frac{\partial V^{*}}{\partial t} + V^{*}\frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial \zeta}{\partial V}\right) = S^{+}_{V}. \tag{28}$$

It can be shown that the adjoint equations, as well as the corresponding boundary conditions, are linear in the adjoint function and independent of the perturbations. As a consequence, for a prescribed response function it is necessary to solve only the non-perturbed direct equations and the corresponding adjoint equations. The sensitivity coefficients for any set of parameters can be calculated a posteriori by means of Eq. (24).

3. Boundary conditions

3.1. Definition of a general hydraulic network

In the following, we shall regard a hydraulic network as a set of one-dimensional pipes interconnected through components. A component can be regarded as a zero-dimensional entity in which a set of relationships involving the state variables corresponding to the pipe boundaries connected to such component are satisfied.

Let us consider a hydraulic network with NT pipes connected through NC components, as shown in Fig. 1; X_{1k} and X_{2k} are correspondingly the positions of the nodes corresponding to the beginning and the end of pipe k, assuming an orientation (denoted by the arrows) assigned to each pipe.

3.2. Generalization to a hydraulic network

By assigning an orientation to each pipe, it is possible to generalize Eq. (26) as

$$P = P_x + P_t, \tag{29}$$

where

$$P_x = -\sum_{k=1}^{NT} \int_{X_{1k}}^{X_{2k}} \left(H^* H_{/i} + V^* V_{/i}\right)\Big|_0^T \mathrm{d}x, \tag{30}$$

$$P_t = -\sum_{k=1}^{NT} \int_0^T \left(H_{/i} g V^* + V_{/i} \frac{a^2}{g} H^*\right)\Big|_{X_{1k}}^{X_{2k}} \mathrm{d}t. \tag{31}$$

Eq. (31) involves a time integration of functions evaluated at the pipe boundaries connected to the components, as shown in Fig. 2. It is convenient to rearrange Eq. (31) by summing over the components as

$$P_t = \sum_{l=1}^{NC} P_{tl}, \tag{32}$$

where

$$P_{tl} = -\sum_{m=1}^{NL} I_{lm} \int_0^T \left(H_{/i} g V^* + V_{/i} \frac{a_m^2}{g} H^*\right)(X_{lm}, t)\, \mathrm{d}t. \tag{33}$$

In Eq. (33) NL is the number of pipes connected to component l. For instance, $NL = 1$ for a constant-level reservoir or for a valve discharging to the atmosphere, $NL = 2$ for a concentrated

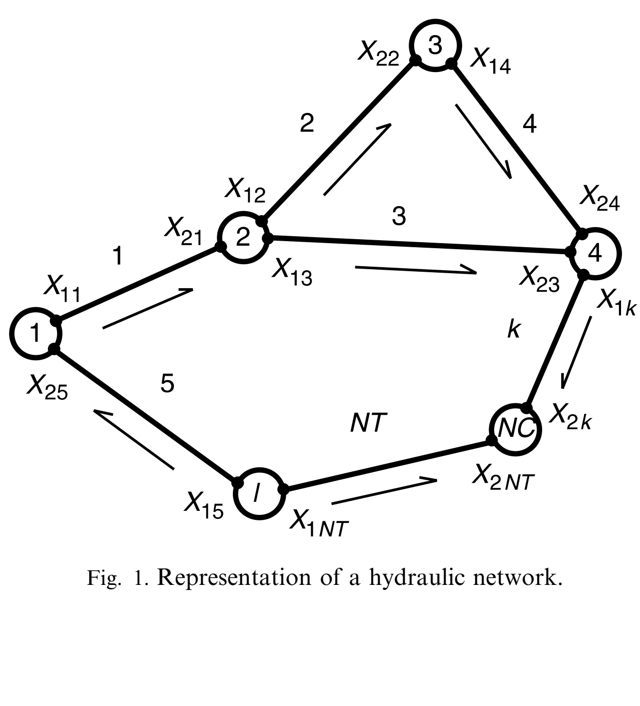

Fig. 1. Representation of a hydraulic network.

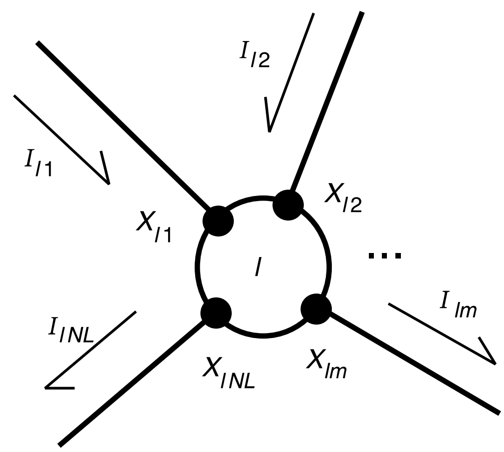

Fig. 2. Representation of a component.

head loss (valve, elbow, etc.), pump or area change, $NL = 3$ for a tee, etc. The index I_{lm} takes into account the orientation of the pipe with respect to the component: $I_{lm} = 1$ if the pipe with boundary X_{lm} is oriented toward the component l, while $I_{lm} = -1$ otherwise.

3.3. Adjoint boundary conditions and bilinear concomitant

The particular form of the adjoint boundary conditions and the bilinear concomitant are determined for each problem, considering that:

(i) The bilinear concomitant must not involve the derived functions, except when evaluated at the initial condition $t = 0$.

(ii) The boundary conditions for the adjoint equations must not involve the derived functions.

3.3.1. Final conditions

From Eq. (30), it can be seen that the adjoint problem must have the following *final conditions* in order to be independent of the derived functions:

$$H^*(x, T) = 0, \tag{34}$$

$$V^*(x, T) = 0. \tag{35}$$

Therefore, the contribution to the bilinear concomitant from Eq. (30) is

$$P_x = \sum_{k=1}^{NT} \int_{X_{1k}}^{X_{2k}} \left(H^* H_{/i} + V^* V_{/i}\right)(x, 0)\,\mathrm{d}x. \tag{36}$$

3.3.2. Component boundary conditions and bilinear concomitant

In [11–13] the adjoint boundary conditions and bilinear concomitant were derived for components connected to a single pipe. In [14,15] the derivation was extended for multipipe components, i.e., components connected to more than one pipe. The derivation of the corresponding adjoint boundary conditions and bilinear concomitant are presented for a constant-level reservoir and a concentrated head loss, the procedure being analogous to other multipipe components.

3.3.2.1. Constant-level reservoir. The component boundary condition for the direct problem comes from the energy conservation equation, namely

$$C_1 \equiv H_1 + \frac{V_1^2}{2g} - I_1 \frac{k_t}{2g} V_1 |V_1| - H_T = 0, \tag{37}$$

where H_1 and V_1 are correspondingly the piezometric head and the velocity at the pipe end connected to the reservoir, H_T is the reservoir liquid level above the pipe, I_1 is the pipe orientation index and k_t is the head loss coefficient, defined as:

$$k_t = \begin{cases} 1 & \text{if } I_1 V_1 \geqslant 0, \\ k_{te} & \text{if } I_1 V_1 < 0. \end{cases} \tag{38}$$

Eq. (33) can be written for a constant-level reservoir as

$$P_{tl} = -I_1 \int_0^T \left(\frac{a_1^2}{g} H_1^* V_{1/i} + g V_1^* H_{1/i} \right) \mathrm{d}t. \tag{39}$$

From Eqs. (18) and (37) we obtain the component boundary condition for the derived problem as

$$C_{1/i} + H_{1/i} + \frac{V_1 V_{1/i}}{g} - I_1 \frac{k_t}{g} V_{1/i} |V_1| = 0. \tag{40}$$

Substituting $H_{1/i}$ from Eq. (40) in Eq. (39) and grouping we obtain

$$P_{tl} = -I_1 \int_0^T V_{1/i} \left(\frac{a_1^2}{g} H_1^* - (V_1 - I_1 k_t |V_1|) V_1^* \right) \mathrm{d}t + I_1 \int_0^T C_{1/i} g V_1^* \, \mathrm{d}t. \tag{41}$$

If we want the bilinear concomitant not to involve the derived function $V_{1/i}$ (except when evaluated at the initial condition $t = 0$) we must choose the following boundary condition for the adjoint function

$$\frac{a_1^2}{g} H_1^* - (V_1 - I_1 k_t |V_1|) V_1^* = 0 \tag{42}$$

or

$$H_1^* + \frac{g}{a_1} \alpha V_1^* = 0, \tag{43}$$

where

$$\alpha = \frac{2g(H_1 - H_T)}{a_1 V_1}. \tag{44}$$

The component contribution to Eq. (41) finally results in

$$P_{tl} = I_1 \int_0^T C_{1/i} g V_1^* \, \mathrm{d}t. \tag{45}$$

3.3.2.2. Valve discharging to a constant piezometric head outlet. Component boundary condition for the direct problem

$$C_1 \equiv H_1 - I_1 \frac{k_v}{2g} V_1 |V_1| - H_v = 0, \tag{46}$$

where H_1 and V_1 are correspondingly the piezometric head and the velocity at the pipe end connected to the valve, H_v is the piezometric head at the valve outlet and k_v is the valve head loss coefficient.

Component boundary condition for the derived problem

$$C_{1/i} + H_{1/i} - I_1 \frac{k_v}{g} V_{1/i}|V_1| = 0. \tag{47}$$

Component boundary condition for the adjoint problem

$$H_1^* + \frac{g}{a_1}\beta V_1^* = 0, \tag{48}$$

where

$$\beta = \frac{2g(H_1 - H_v)}{a_1 V_1}. \tag{49}$$

Component contribution to Eq. (33)

$$P_{tl} = I_1 \int_0^T C_{1/i} g V_1^* \, \mathrm{d}t. \tag{50}$$

3.3.2.3. Concentrated head loss. Components like partially open valves, elbows, orifice plates, etc., connecting pipes of the same diameter can be modelled as concentrated head losses. The component boundary conditions for the direct problem come from the energy conservation equation and the mass conservation equation, namely:

$$C_1 \equiv H_1 - H_2 - I_1 \frac{k_p}{2g} V_1|V_1| = 0, \tag{51}$$

$$C_2 \equiv I_1 V_1 + I_2 V_2 = 0, \tag{52}$$

where k_p is the head loss coefficient.

Eq. (33) can be written for a concentrated head loss as

$$P_{tl} = -I_1 \int_0^T \left(H_{1/i} g V_1^* + V_{1/i} \frac{a_1^2}{g} H_1^* \right) \mathrm{d}t - I_2 \int_0^T \left(H_{2/i} g V_2^* + V_{2/i} \frac{a_2^2}{g} H_2^* \right) \mathrm{d}t. \tag{53}$$

From Eqs. (18), (51) and (52) we obtain the component boundary condition for the derived problem as:

$$C_{1/i} + H_{1/i} - H_{2/i} - I_1 \frac{k_p}{2g} V_{1/i}|V_1| = 0, \tag{54}$$

$$I_1 V_{1/i} + I_2 V_{2/i} = 0. \tag{55}$$

Substituting $H_{2/i}$ and $V_{2/i}$ from Eqs. (54) and (55) in Eq. (53) and cancelling the resulting terms in $H_{1/i}$ and $V_{1/i}$, we have for the adjoint boundary conditions:

$$H_1^* - \left(\frac{a_2}{a_1} \right)^2 H_2^* + \frac{g}{a_1}\gamma V_1^* = 0, \tag{56}$$

$$I_1 V_1^* + I_2 V_2^* = 0, \tag{57}$$

where

$$\gamma = \frac{2g(H_1 - H_2)}{a_1 V_1}. \tag{58}$$

The component contribution to Eq. (53), after taking into account the boundary conditions of Eqs. (56) and (57), finally results in

$$P_{tl} = I_1 \int_0^T C_{1/i} g V_1^* \, \mathrm{d}t. \tag{59}$$

3.3.2.4. Area change $(D_2 \geqslant D_1)$. Direct boundary conditions:

$$C_1 \equiv H_1 - H_2 + \frac{1}{2g}\left(V_1^2 - V_2^2\right) - I_1 \frac{k_p}{2g} V_1 |V_1| = 0, \tag{60}$$

$$C_2 \equiv I_1 D_1^2 V_1 + I_2 D_2^2 V_2 = 0, \tag{61}$$

where k_p is the head loss coefficient, calculated as [16]:

$$k_p = \begin{cases} \left[1 - \left(\dfrac{D_1}{D_2}\right)^2\right]^2 & \text{if } I_1 V_1 \geqslant 0, \\ 0.42\left[1 - \left(\dfrac{D_1}{D_2}\right)^2\right] & \text{if } I_1 V_1 < 0. \end{cases} \tag{62}$$

Adjoint boundary conditions:

$$H_1^* - \left(\frac{a_2}{a_1}\right)^2 \left(\frac{D_1}{D_2}\right)^2 H_2^* + \frac{g}{a_1} \eta V_1^* = 0, \tag{63}$$

$$I_1 V_1^* + I_2 V_2^* = 0, \tag{64}$$

where

$$\eta = \frac{2g(H_1 - H_2)}{a_1 V_1}. \tag{65}$$

Final contribution to the bilinear concomitant

$$P_{tl} = \int_0^T \left[I_1 C_{1/i} g V_1^* + \frac{C_{2/i}}{D_2^2} \left(\frac{a_2^2}{g} H_2^* - V_2 V_2^* \right) \right] \mathrm{d}t. \tag{66}$$

3.3.2.5. Tee without head losses. Direct boundary conditions:

$$C_1 \equiv H_1 - H_2 + \frac{1}{2g}(V_1^2 - V_2^2) = 0, \tag{67}$$

$$C_2 \equiv H_1 - H_3 + \frac{1}{2g}(V_1^2 - V_3^2) = 0, \tag{68}$$

$$C_3 \equiv I_1 D_1^2 V_1 + I_2 D_2^2 V_2 + I_3 D_3^2 V_3 = 0. \tag{69}$$

Adjoint boundary conditions:

$$H_1^* - \left(\frac{a_2}{a_1}\right)^2 \left(\frac{D_1}{D_2}\right)^2 H_2^* - \frac{g}{a_1}\frac{V_1}{a_1} V_1^* + \left(\frac{D_1}{D_2}\right)^2 \frac{g}{a_1}\frac{V_2}{a_1} V_2^* = 0, \tag{70}$$

$$H_1^* - \left(\frac{a_3}{a_1}\right)^2 \left(\frac{D_1}{D_3}\right)^2 H_3^* - \frac{g}{a_1}\frac{V_1}{a_1} V_1^* + \left(\frac{D_1}{D_3}\right)^2 \frac{g}{a_1}\frac{V_3}{a_1} V_3^* = 0, \tag{71}$$

$$I_1 V_1^* + I_2 V_2^* + I_3 V_3^* = 0. \tag{72}$$

Since Eqs. (67) and (68) have no parameters (except the acceleration of gravity), the final contribution to the bilinear concomitant is

$$P_{tl} = \int_0^T \frac{C_{3/i}}{D_1^2}\left(\frac{a_1^2}{g}H_1^* + I_1I_2V_1V_2^* + I_1I_3V_1V_3^*\right)\mathrm{d}t. \tag{73}$$

3.3.3. Sensitivity coefficients for parameters influencing the initial condition

A problem still remains, which is the calculation of Eq. (36). For the sake of consistency with the adjoint problem, it would be convenient to perform this calculation without choosing a priori any parameter. Eq. (36) can be written as

$$P_x = -\sum_{k=1}^{NT}\int_{X_{1k}}^{X_{2k}}\left[H^*(x,0)\tilde{H}_{/i} + V^*(x,0)\tilde{V}_{/i}\right]\mathrm{d}x, \tag{74}$$

where the superscript $\sim$ denotes a value for the initial condition.

Let us consider the following perturbative problem corresponding to the waterhammer equations *in steady state*, characterized by the response functional $\tilde{R}$

$$\tilde{R} = \langle\tilde{\boldsymbol{S}}^+\cdot\tilde{\boldsymbol{f}}\rangle = \sum_{k=1}^{NT}\int_{X_{1k}}^{X_{2k}}\left(\tilde{S}_H^+\tilde{H} + \tilde{S}_V^+\tilde{V}\right)\mathrm{d}x. \tag{75}$$

According to the definition of the adjoint operator,

$$\langle\tilde{\boldsymbol{f}}_{/i}\cdot\tilde{\boldsymbol{S}}^+\rangle = \langle\tilde{\boldsymbol{f}}\cdot\tilde{\boldsymbol{S}}(i)\rangle + \tilde{P}\left(\tilde{\boldsymbol{f}}^*,\tilde{\boldsymbol{f}}_{/i}\right), \tag{76}$$

where $\tilde{\boldsymbol{S}}(i)$ is also a known vector function coming from the direct steady-state equations.

Comparing Eqs. (76) with Eq. (74), it can be seen that the left-hand side of Eq. (76) is equal to $-P_x$ if the vector weight function is chosen as

$$\tilde{\boldsymbol{S}}^+ = [H^*(x,0), V^*(x,0)]^{\mathrm{T}}. \tag{77}$$

In summary, the calculation of any sensitivity coefficient can be performed by solving sequentially two perturbative problems: the first one is non-steady and was described in Section 2, while the second one is steady, with an appropriate selection of a weight function, namely Eq. (77), coming from the unsteady perturbative problem. The adjoint functions $\boldsymbol{f}^*$ and $\tilde{\boldsymbol{f}}^*$ are independent of the parameters to be perturbed.

Usually, the initial condition for the direct problem corresponds to the system in a steady state. The calculation of the steady adjoint function can be performed with the aid of Eqs. (27) and (28), in which the source terms are given by Eq. (77), starting from any final condition and letting the system evolve to the steady state.

The contribution corresponding to any component to the bilinear concomitant $\tilde{P}$ can be calculated with a methodology similar to the one outlined in Section 3.3.

4. Response functionals

4.1. Some response functionals

From Eqs. (27) and (28) it can be observed that the solution of the adjoint equation involves the knowledge of the solution of the corresponding direct equations and the definition of the source term $\boldsymbol{S}^{+}$. The instantaneous values for the piezometric head and velocity can be chosen as response functionals by means of suitable distribution functions.

4.1.1. Instantaneous piezometric head

Let us consider the weight function

$$\boldsymbol{S}^{+} = [\delta(\boldsymbol{r} - \boldsymbol{r_0}), 0]. \tag{78}$$

From Eq. (4), we get

$$R \equiv H(\boldsymbol{r_0}) = H(x_0, T). \tag{79}$$

4.1.2. Instantaneous velocity

Let us consider the weight function

$$\boldsymbol{S}^{+} = [0, \delta(\boldsymbol{r} - \boldsymbol{r_0})]. \tag{80}$$

In this case, we get

$$R \equiv V(\boldsymbol{r_0}) = V(x_0, T). \tag{81}$$

4.2. Analysis of the adjoint function

According to Eqs. (78) and (80), it can be seen that if the instantaneous values of the piezometric head or velocity are chosen as response functionals, then the source terms (correspondingly S_H^{+} or S_V^{+}) are impulse (delta) functions acting at the observation point at the final time T; as a consequence, the source terms corresponding to the adjoint equations are zero for all times $t < T$. It was shown [11,14,15] that due to the hyperbolic nature of the adjoint equations, these delta-functions travel along the hydraulic network with the propagation speed, changing their powers due to the influence of the friction term and due to the transmissions and reflections undergone as they reach the different components.

4.2.1. Reflection and transmission coefficients

Neglecting the gravity and friction terms and considering no source terms, the general solution of Eqs. (27) and (28) can be written as:

$$H^{*} = f^{*}\left(t - \frac{x}{a}\right) + F^{*}\left(t + \frac{x}{a}\right), \tag{82}$$

$$V^{*} = \frac{a}{g}\left[f^{*}\left(t - \frac{x}{a}\right) - F^{*}\left(t + \frac{x}{a}\right)\right], \tag{83}$$

where (see Fig. 3), for *decreasing* times, f^{*} is a wave propagating in the direction of decreasing positions, while F^{*} is a wave propagating in the direction of increasing positions. Eqs. (82) and

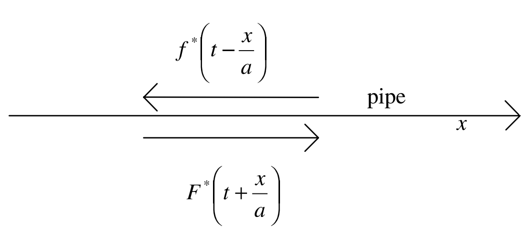

Fig. 3. Adjoint wave propagation for decreasing times.

(83) are analogous expressions to the ones obtained in [17] for the direct problem. Therefore, it is possible to derive the reflection and transmission coefficients for any component in the adjoint problem.

The reflection and transmission coefficients for the adjoint piezometric head and velocity are defined correspondingly as

$$r_H^* = \frac{(H^*)_\mathrm{r}}{(H^*)_\mathrm{i}}, \tag{84}$$

$$t_H^* = \frac{(H^*)_\mathrm{t}}{(H^*)_\mathrm{i}}, \tag{85}$$

$$r_V^* = \frac{(V^*)_\mathrm{r}}{(V^*)_\mathrm{i}}, \tag{86}$$

$$t_V^* = \frac{(V^*)_\mathrm{t}}{(V^*)_\mathrm{i}} \tag{87}$$

where $(H^*)_\mathrm{r}$, $(H^*)_\mathrm{t}$ and $(H^*)_\mathrm{i}$ are correspondingly the reflected, transmitted and incident adjoint piezometric heads, while $(V^*)_\mathrm{r}$, $(V^*)_\mathrm{t}$ and $(V^*)_\mathrm{i}$ are correspondingly the reflected, transmitted and incident adjoint velocities. As in Section 3.3.2, the derivation is outlined for a constant-level reservoir and for a concentrated head loss, the procedure being analogous to other multipipe components.

4.2.1.1. Constant-level reservoir. Assuming $I_1 = -1$, we have

$$(H^*)_\mathrm{i} = f_1^*, \tag{88}$$

$$(H^*)_\mathrm{r} = F_1^*. \tag{89}$$

From Eq. (83) we get:

$$(V^*)_\mathrm{i} = \frac{a_1}{g} f_1^*, \tag{90}$$

$$(V^*)_\mathrm{r} = -\frac{a_1}{g} F_1^*. \tag{91}$$

Replacing Eqs. (88) and (89) in Eq. (82), then replacing Eqs. (90) and (91) in Eq. (83) and taking into account Eq. (83) we have:

$$F_1^* = -\frac{1+\alpha}{1-\alpha} f_1^*. \tag{92}$$

Taking into account Eqs. (84), (86) and Eqs. (88)–(91) we finally obtain:

$$r_H^* = -\frac{1+\alpha}{1-\alpha}, \tag{93}$$

$$r_V^* = -r_H^*. \tag{94}$$

4.2.1.2. Valve discharging to a constant piezometric head outlet. Assuming $I_1 = 1$, we have:

$$r_H^* = -\frac{1-\beta}{1+\beta}, \tag{95}$$

$$r_V^* = -r_H^*. \tag{96}$$

4.2.1.3. Concentrated head loss. Assuming $I_1 = 1$, $I_2 = -1$ and that the incident wave comes from side 1, we have:

$$(H_1^*)_{\mathrm{i}} = F_1^*, \tag{97}$$

$$(H_1^*)_{\mathrm{r}} = f_1^*, \tag{98}$$

$$(H_1^*)_{\mathrm{t}} = F_2^*, \tag{99}$$

$$f_2^* = 0. \tag{100}$$

From Eq. (83) applied to sides 1 and 2 we get:

$$(V_1^*)_{\mathrm{i}} = -\frac{a_1}{g} F_1^*, \tag{101}$$

$$(V_1^*)_{\mathrm{r}} = \frac{a_1}{g} f_1^*, \tag{102}$$

$$(V_1^*)_{\mathrm{t}} = -\frac{a_2}{g} F_2^*. \tag{103}$$

Taking into account Eqs. (82) and (83) and the boundary conditions given by Eqs. (56) and (57), we obtain:

$$f_1^* = \frac{(a_2/a_1) + \gamma - 1}{(a_2/a_1) + \gamma + 1} F_1^*, \tag{104}$$

$$F_2^* = \frac{2(a_2/a_1)}{(a_2/a_1) + \gamma + 1} F_1^*. \tag{105}$$

From Eq. (84)–(87) we finally obtain:

$$r_H^* = \frac{(a_2/a_1) + \gamma - 1}{(a_2/a_1) + \gamma + 1}, \tag{106}$$

$$t_H^* = \frac{2(a_1/a_2)}{(a_2/a_1) + \gamma + 1}, \tag{107}$$

$$r_V^* = -r_H^*, \tag{108}$$

$$t_V^* = \frac{a_2}{a_1} t_H^*. \tag{109}$$

5. The discrete problem

5.1. Method of characteristics

It can be shown that the direct and adjoint equations are hyperbolic, so they can be transformed into ordinary differential equations along characteristic curves defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a \quad (\text{curve } C^+), \tag{110}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -a \quad (\text{curve } C^-). \tag{111}$$

For the direct equations, we have [8]:

$$\frac{DH}{Dt} + \frac{a}{g}\frac{DV}{Dt} + V\sin\theta + \frac{a\zeta V|V|}{2gD} = 0 \quad \text{along } C^{+}, \tag{112}$$

$$\frac{DH}{Dt} - \frac{a}{g}\frac{DV}{Dt} + V\sin\theta - \frac{a\zeta V|V|}{2gD} = 0 \quad \text{along } C^{-}. \tag{113}$$

Correspondingly, for the adjoint equations we have:

$$\frac{DH^*}{Dt} + \frac{g}{a}\frac{DV^*}{Dt} - H^*\frac{g}{a}\sin\theta - V^*\frac{g}{a}\frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial\zeta}{\partial V}\right) + S_H^{+} + \frac{g}{a}S_V^{+} = 0 \text{ along } C^{+}, \tag{114}$$

$$\frac{DH^*}{Dt} - \frac{g}{a}\frac{DV^*}{Dt} + H^*\frac{g}{a}\sin\theta + V^*\frac{g}{a}\frac{|V|}{D}\left(\zeta + \frac{V}{2}\frac{\partial\zeta}{\partial V}\right) + S_H^{+} - \frac{g}{a}S_V^{+} = 0 \text{ along } C^{-}. \tag{115}$$

5.2. Discretization

The characteristic curves are the same for the direct and adjoint equations and have a constant slope. Thus, it is possible to discretize regularly the plane x–t into nodes and cycles, as in finite difference schemes, and perform the integration along the characteristics.

The direct equations have initial conditions, so the integration proceeds *forward* in time. For an inner node, the values of H and V after a time-step are calculated by simultaneously solving finite difference approximations of the characteristic equations (112) and (113). For a node connected to a spatial boundary condition, it is necessary to solve simultaneously the appropriate characteristic equation and the corresponding boundary condition.

The strategy used in the integration of the adjoint equations is similar to the one detailed above. However, it will be shown later that the adjoint equations have *final* conditions; thus, the integration must proceed *backward* in time.

The direct problem was programmed in the computer code Water Hammer Analysis in Tubes (WHAT) [18]. With this code, any hydraulic network can be built by connecting different components (tanks, valves, pumps, tees, etc.) through pipes. The adjoint problem was programmed for different response functionals in the computer code ADWHAT [11,14], keeping the same philosophy.

It is worth noting that the numerical solution of the adjoint problem involved the propagation of delta-functions, which required modifications to the usual method of characteristics in order to avoid "checkerboard" effects in the solution fields [11]. The details of these modifications are, however, out of the scope of the present work.

6. Application example

Let us consider the problem of a single pipe connected at the end $x = 0$ to a constant-level tank, while the end $x = X$ is connected to a valve discharging to atmosphere, as shown in Fig. 4.

6.1. Direct equations

The direct boundary conditions for this case come from Eq. (37) (in which $I_1 = -1$) and Eq. (46) (in which $I_1 = 1$):

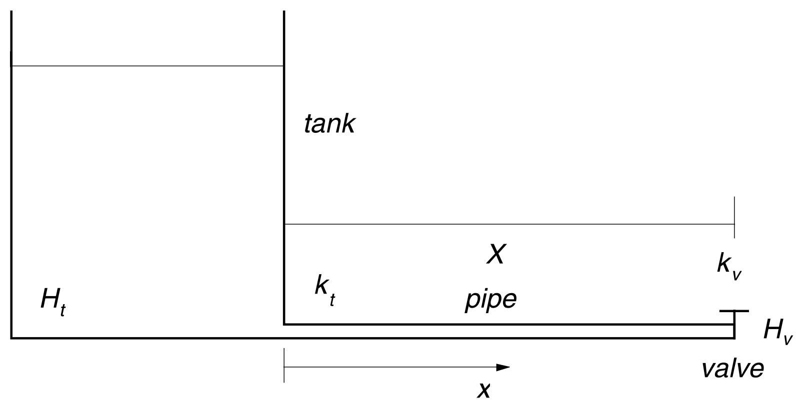

Fig. 4. Hydraulic system.

$$C_1 \equiv H + \frac{1}{2g}\left(V^2 + k_t V|V|\right) - H_t = 0 \quad \text{at } x = 0, \tag{116}$$

$$C_2 \equiv H - \frac{1}{2g} k_v V|V| - H_v = 0 \quad \text{at } x = X, \tag{117}$$

$$C_3 \equiv H - \tilde{H} = 0 \quad \text{at } t = 0, \tag{118}$$

$$C_4 \equiv V - \tilde{V} = 0 \quad \text{at } t = 0, \tag{119}$$

where C_1 and C_2 represent the boundary conditions at the tank and at the valve, while C_3 and C_4 represent the initial conditions.

6.2. Derived equations

The derived boundary conditions result in:

$$H_{/i} + V_{/i}\frac{1}{g}(V + k_t|V|) + C_{1/i} = 0 \quad \text{at } x = 0, \tag{120}$$

$$H_{/i} - V_{/i}\frac{k_v}{g}|V| + C_{2/i} = 0 \quad \text{at } x = X, \tag{121}$$

$$H_{/i} - \tilde{H}_{/i} = 0 \quad \text{at } t = 0, \tag{122}$$

$$V_{/i} - \tilde{V}_{/i} = 0 \quad \text{at } t = 0. \tag{123}$$

6.3. Adjoint equations and bilinear concomitant

The adjoint boundary conditions result in:

$$C_1^* \equiv \frac{a^2}{g}H^* - (V + k_t|V|)V^* = 0 \quad \text{at } x = 0, \tag{124}$$

$$C_2^* \equiv \frac{a^2}{g}H^* + k_v|V|V^* = 0 \quad \text{at } x = X, \tag{125}$$

$$C_3^* \equiv H^* = 0 \quad \text{at } t = 0, \tag{126}$$

$$C_4^* \equiv V^* = 0 \quad \text{at } t = 0. \tag{127}$$

Note that the *initial* conditions in the direct problem have been transformed into *final* conditions in the adjoint problem.

The bilinear concomitant can be calculated as:

$$P = \int_0^X \left[H^*(x,0)\tilde{H}_{/i}(x) + V^*(x,0)\tilde{V}_{/i}(x) \right] \mathrm{d}x + \int_0^T C_{2/i}(X,t) g V^*(X,t)\,\mathrm{d}t - \int_0^T C_{1/i}(0,t) g V^*(0,t)\,\mathrm{d}t. \tag{128}$$

6.4. Definition of the sensitivity problem

We consider the hydraulic system in a steady state for $t \leqslant 0$. For $t \geqslant 0$ the valve is operated in such a way that the friction coefficient changes linearly from k_{vi} to k_{vf} in a time interval τ, as shown in Fig. 5:

$$k_v = \begin{cases} k_{vi} + \dfrac{k_{vf} - k_{vi}}{\tau} & \text{for } 0 \leqslant t \leqslant \tau, \\ k_{vf} & \text{for } t > \tau. \end{cases} \tag{129}$$

The following constants were chosen: $H_t = 13$ m, $H_t = 10$ m, $X = 30$ m, $D = 2.54 \times 10^{-2}$ m, $\varepsilon = 1 \times 10^{-4}$ m, $k_{vi} = 0$, $k_{vf} = 1 \times 10^{-3}$, $\tau = 8.35 \times 10^{-2}$ s and $k_{te} = 0.5$. The fluid is water, with $\upsilon = 1 \times 10^{-6}$ m^2/s. The resulting propagation speed is $a = 1437$ m/s. The pipe was discretized into 11 nodes, resulting in $\Delta x = 3$ m, $\Delta t = \Delta x / a = 2.087 \times 10^{-3}$ s. We are interested in the sensitivity coefficients defined by Eqs. (24) and (26) due to variations of different parameters.

6.5. Some results

Once the direct problem for the given example was numerically solved with the code WHAT, six related adjoint problems were solved with the code ADWHAT. These six cases have different response functionals defined as:

(a) $R = H(0,T)$, head at the tank outlet (node 1);
(b) $R = H(X/2,T)$, head at the middle of the pipe (node 6);
(c) $R = H(X,T)$, head at the valve inlet (node 11);
(d) $R = V(0,T)$, velocity at the tank outlet (node 1);
(e) $R = H(X/2,T)$, velocity at the middle of the pipe (node 6);
(f) $R = V(X,T)$, velocity at the valve inlet (node 11).

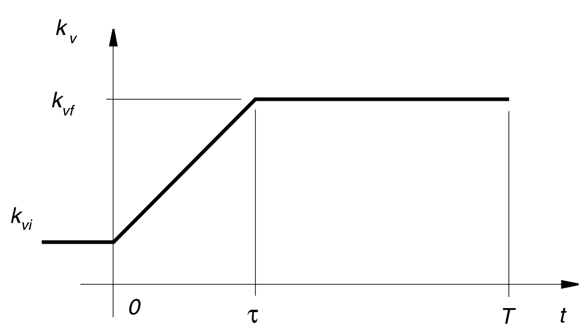

Fig. 5. Time variation of the valve friction coefficient.

As an example of the direct solution, the evolution of H and V during the transient at the three selected locations (nodes 1, 6 and 11) is shown in Figs. 6–8.

For each one of the previously defined six adjoint cases, the sensitivity coefficients due to changes in the parameter k_{vf} and the closure time τ were calculated, by means of the computer code SANWHAT. Two different values of the reference time T ($T = \tau/2$ and $T = 20\tau$) were chosen. Consequently, 24 different sensitivity coefficients have been evaluated for each parameter. They are tabulated in Tables 1 and 2 along with the sensitivities evaluated by obtaining direct solutions with the selected perturbed parameters, using the code WHAT.

The closure time has been chosen equal to the period of the perturbation ($4X/a$). For $T = \tau/2$ we get the maximum value of the piezometric head at the valve, while for $T = 20\tau$ a new steady state is achieved.

It is important to notice that the parameters chosen to investigate the sensitivity neither influence the direct equations nor the initial conditions, so in Eq. (24) $\boldsymbol{S}^{+}_{/i} = \boldsymbol{S}(i) = \boldsymbol{0}$ and in Eq. (128)

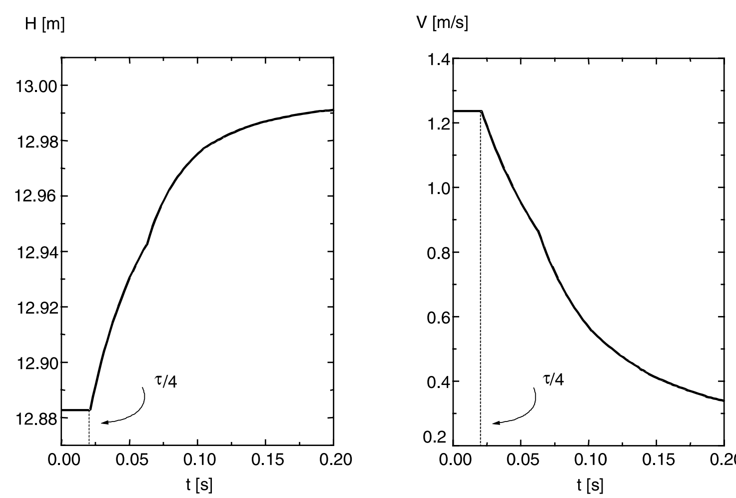

Fig. 6. Evolution of H and V at the tank outlet (node 1).

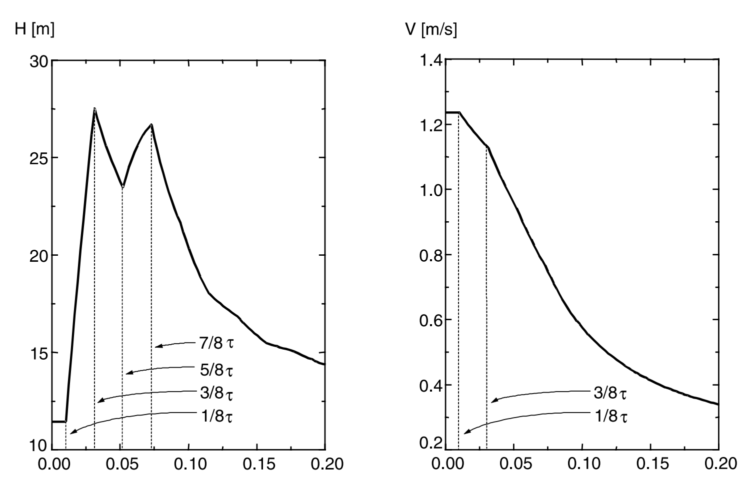

Fig. 7. Evolution of H and V at the middle of the pipe (node 6).

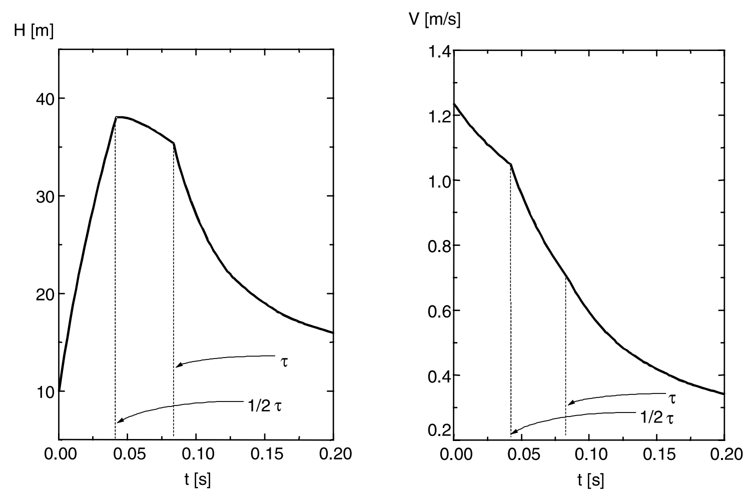

Fig. 8. Evolution of H and V at the valve inlet (node 11).

Table 1
Sensitivity coefficients for $T = \tau/2$

Node	1		6		11	
	WHAT	SANWHAT	WHAT	SANWHAT	WHAT	SANWHAT
$\frac{\delta H}{\delta k_{vf}}$ (m)	2.83×10^{-5}	2.83×10^{-5}	9.65×10^{-3}	9.64×10^{-3}	2.06×10^{-2}	2.06×10^{-2}
$\frac{\delta V}{\delta k_{vf}}$ (m/s)	-1.81×10^{-4}	-1.81×10^{-4}	-1.71×10^{-4}	-1.71×10^{-4}	-1.38×10^{-4}	-1.38×10^{-4}
$\frac{\delta H}{\delta \tau}$ (m)	−0.34	−0.33	−115.6	−115.5	−247.0	−247.1
$\frac{\delta V}{\delta \tau}$ (m/s)	2.170	2.171	2.050	2.051	0.65	0.66

$\tilde{H}_{/i} = \tilde{V}_{/i} = 0$. Since the parameters chosen are only related to the boundary condition at the valve, namely Eq. (117), we finally get for the sensitivity coefficients:

$$\frac{\delta R}{\delta k_{vf}} = \int_0^T \frac{\partial C_2}{\partial k_{vf}} g V^*(X,t)\,\mathrm{d}t, \tag{130}$$

where

$$\frac{\partial C_2}{\partial k_{vf}} = \begin{cases} 0 & \text{for } t < 0, \\ -\dfrac{1}{2g} V|V| \dfrac{t}{\tau} & \text{for } 0 \leqslant t \leqslant \tau, \\ -\dfrac{1}{2g} V|V| & \text{for } t > \tau, \end{cases} \tag{131}$$

$$\frac{\delta R}{\delta \tau} = \int_0^T \frac{\partial C_2}{\partial \tau} g V^*(X,t)\,\mathrm{d}t, \tag{132}$$

Table 2
Sensitivity coefficients for $\tau = 20\tau$

Node	1		6		11	
	WHAT	SANWHAT	WHAT	SANWHAT	WHAT	SANWHAT
$\frac{\delta H}{\delta k_{vf}}$ (m)	4.11×10^{-6}	4.14×10^{-6}	6.15×10^{-5}	5.27×10^{-5}	1.19×10^{-4}	1.01×10^{-4}
$\frac{\delta V}{\delta k_{vf}}$ (m/s)	-1.13×10^{-4}	-1.14×10^{-4}	-1.13×10^{-4}	-1.14×10^{-4}	-1.13×10^{-4}	-1.14×10^{-4}
$\frac{\delta H}{\delta \tau}$ (m)	-6.0×10^{-8}	-7.1×10^{-8}	-8.2×10^{-5}	-8.4×10^{-5}	-8.2×10^{-5}	-8.7×10^{-5}
$\frac{\delta V}{\delta \tau}$ (m/s)	1.8×10^{-6}	2.0×10^{-6}	1.8×10^{-6}	2.0×10^{-6}	3.3×10^{-6}	3.6×10^{-6}

where

$$\frac{\partial C_2}{\partial \tau} = \begin{cases} 0 & \text{for } t < 0, \\ \dfrac{1}{2g}\,\dfrac{k_{vf} - k_{vi}}{\tau}\,V|V|\dfrac{t}{\tau} & \text{for } 0 \leqslant t \leqslant \tau, \\ 0 & \text{for } t > \tau. \end{cases} \tag{133}$$

From Tables 1 and 2 it can be observed that the agreement between the results obtained from the codes WHAT and SANWHAT is excellent for short observation times. For the steady state, there are slight discrepancies due primarily to the treatment of the friction term in the direct equations when the method of characteristics is applied.

In Fig. 9, the adjoint solutions H^* and V^* for case (a) at some selected cycles are shown. This figure is only intended to show the kind of travelling delta-like waves that are typical of the

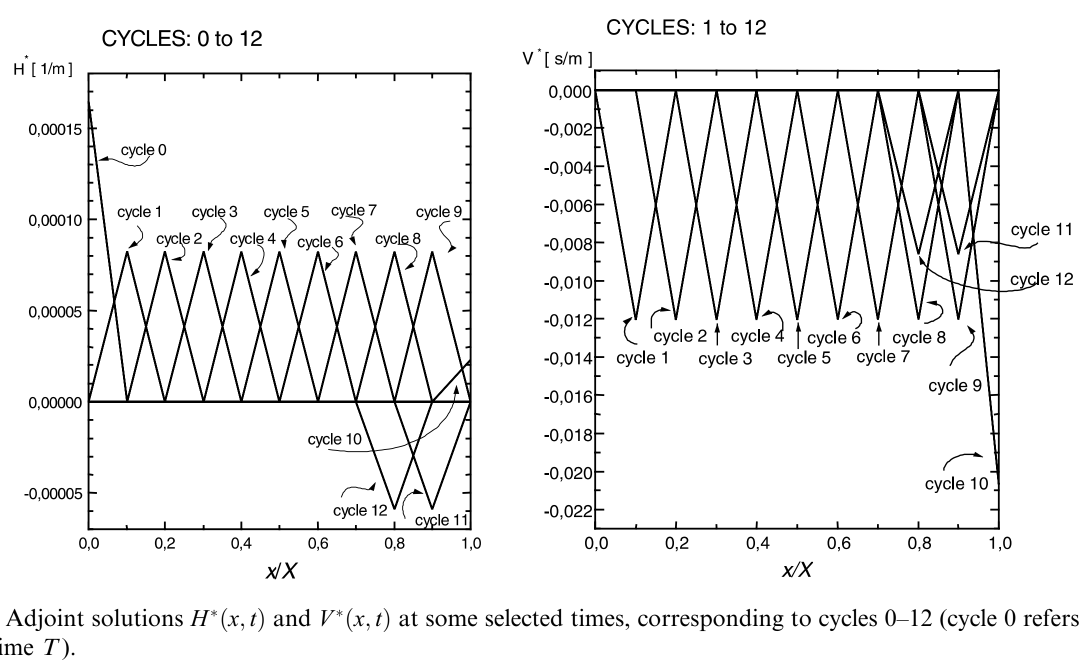

Fig. 9. Adjoint solutions $H^*(x,t)$ and $V^*(x,t)$ at some selected times, corresponding to cycles 0–12 (cycle 0 refers to the final time T).

adjoint waterhammer problems. In Fig. 9 the area of each triangle indicates the power of the delta-function.

7. Conclusions

The development of the sensitivity theory by the differential method for a general waterhammer problem was outlined. The adjoint equations and the general form of the bilinear concomitant were obtained. The methodology was applied to a simple problem, showing excellent agreement between the sensitivity coefficients calculated with the differential method and the ones obtained via the solution of many perturbed direct problems. The authors hope that this paper will encourage the use of perturbative methods for sensitivity analysis in different areas of Engineering Science.

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