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1. Introduction.

The problem of the determination of the disintegration constant of a radio-isotope when the variation of counts with time is known, was considered by Peierls[1]. This author treated a simplified case, i.e., when the intervals of counts are small compared with the mean life of the phenomenon.

In this paper this last assumption is discarded and an exact theory is presented. We obtain a formula that allows the determination of the mean life when counts are taken during equal intervals of time equally spaced and not necessarily small when compared with the mean life. An expression for the error involved in this determination of mean life is also obtained. Finally, a simplified expression for the calculation of mean life for the case of small intervals of counting, separated by intervals of finite width, is presented.

2. Mean life determination: Adjacent counting intervals.

Suppose that we have a source emitting particles that are counted during k intervals of time of width Δ, T being the total counting time and n_v the total number of counts in the v^{th} interval of counting.

If the source decays with a mean life τ, the probabili ty of observing a particle between t and t+dt is given by:

$$W(t) = \frac{1}{\tau} e^{-t/\tau} dt \tag{1}$$

or

$$W(t) = \lambda e^{-\lambda t} dt \tag{2}$$

where

$$\lambda = \frac{1}{\tau} \tag{3}$$

The probability of observing a particle in the v^{th} interval is:

$$W_v = \int_{t_{v-1}}^{t_v} W(t)\, dt = e^{-\lambda t_v} (e^{\lambda \Delta} - 1) \tag{4}$$

This expression gives the probability of observing a particle in the v^{th} interval, when it is certain that it will be observed in the interval from zero to infinity. This can seen nothing that:

$$\int_0^{\infty} W(t)\, dt = 1 \tag{5}$$

If we consider one group of m particles that we know were observed in k intervals between zero and T, the probability that one of those will be observed in the v^{th} interval is no longer given by (4) but by $\overline{W}_v$, with:

$$\overline{W}_v = \frac{W_v}{W} \tag{6}$$

where

$$W = \sum_{v=1}^{k} W_v = 1 - e^{-\lambda T} . \tag{7}$$

W is the probability of observing a particle in the interval from zero to T, when it is only known that it is certain to observe it in the interval from zero to infinity.

Indeed, let us consider one group of N particles that are observed in the total time interval, that is, zero to infinity. The most probable number of those which are observed in the vth interval will be NW_v while the most probable number of these which will occur between zero and T will be NW.

Hence, the relative frequency of the particles which fall in the vth interval, and belong to the group that fall between zero and T will be:

$$\frac{N.W_v}{N.W} = \frac{W_v}{W} = \bar{W}_v \qquad (8)$$

Hence,

$$\bar{W}_v = \frac{e^{-\lambda t_v} \left[e^{\lambda\Delta} - 1 \right]}{1 - e^{-\lambda T}} \qquad (9)$$

Now, consider a distribution of m particles, observed in k intervals between zero and T. If this distribution is characterized by n_1, n_2, ... n_k, the probability of its occurrence will be give by:

$$P\,(m, T, \lambda\,;\, n_v) = \frac{m!}{n_1!\; n_2!\; \ldots\; n_k!} \left(\bar{W}_1\right)^{n_1} \ldots \left(\bar{W}_k\right)^{n_k} \qquad (10)$$

with

$$m = \sum_{v=1}^{k} n_v \qquad (11)$$

Using (6) and (11) we can write (10) as:

$$P\ (m,T,\lambda\ ;\ n_v) = \frac{m!}{n_1!\ n_2!\ \dots\ n_k!}(W_1)^{n_1}\dots(W_k)^{n_k}\ \left(\frac{1}{W}\right)^m \tag{12}$$

The same result will be obtained writing:

$$P\ (m,\ T,\lambda\ ;\ n_v) = C\ \frac{m!}{n_1!\ n_2!\ \dots\ n_k!}(W_1)^{n_1}\ \dots\ (W_k)^{n_k} \tag{13}$$

and determining C by the condition of normalization:

$$\sum_{n_1+n_2+\dots+n_k=m} P\ (m,\ T,\lambda\ ;\ n_v) = 1 \tag{14}$$

By the "maximum" likehood principle", λ will be given by the condition of maximizing the value of P or of $\ln P$.

From (12), we have:

$$\ln P = \ln m! - \sum_{v=1}^{k} \ln n_v! + \sum_{v=1}^{k} n_v \ln W_v - m \ln W \tag{15}$$

From (7) and (4), we obtain:

$$\ln P = A - \frac{ms}{\tau} + m \ln \left(e^{\lambda\Delta} - 1\right) - m \ln \left(1 - e^{-\lambda T}\right) \tag{16}$$

with A a constant, independent of λ.

For maximum $\ln P$, we shall have:

$$\frac{\partial}{\partial\lambda}\ (\ln P) = 0 \tag{17}$$

And the equation that we obtain is:

$$s - \frac{\Delta}{1 - e^{-\lambda\Delta}} + \frac{T}{e^{\lambda T} - 1} = 0 \qquad (18)$$

with

$$s = \frac{1}{m} \sum_{\nu = 1}^{k} t_\nu n_\nu \qquad (19)$$

Since: $\lambda = \frac{1}{\tau}$ we can write (19) as:

$$s - \frac{\Delta}{1 - e^{-\Delta/\tau}} + \frac{T}{e^{T/\tau} - 1} = 0 \qquad (20)$$

This expression allows the calculation of τ, once the values of T and Δ are known and s is calculated from (19). In the appendix a practical method for solving the transcendental equations of this type is introduced.

If $\Delta \ll \tau$

$$1 - e^{-\Delta/\tau} \simeq \Delta/\tau \qquad (21)$$

$$\frac{\Delta}{1 - e^{-\Delta/\tau}} \simeq \tau \qquad (22)$$

and (20) reduce to the Peierls' approximate formula:

$$\tau - s = \frac{T}{e^{T/\tau} - 1} \qquad (23)$$

It is interesting to note that the application of Peierls' formula finite intervals with width Δ, requires the calculation of the position, in the v^{th} interval, of the value t_v that appears in the expression of s; and such a calculation involves a approximate value of τ in the correction terms.

In the exact equation (20) s is calculated through(19), where t_v means exactly the end of v^{th} interval.

3. Mean life determination. Non adjacent counting intervals.

If Δ is the counting interval and δ is the time separation between intervals of counting, then:

$$t_n = n\Delta + (n - 1)\delta \tag{24}$$

$$t_n = n(\Delta + \delta) - \delta \tag{25}$$

$$T = (\Delta + \delta) . k \tag{26}$$

In this case, the formula that gives W is:

$$W = \sum_{n=1}^{k} w_n \tag{27}$$

where

$$w_n = e^{-\lambda t_n} \left[e^{\lambda\Delta} - 1 \right] \tag{28}$$

Taking (25) and summing the geometric series, we will have:

$$W = \frac{\left[1 - e^{-\lambda\Delta}\right]\left[1 - e^{-\lambda T}\right]}{\left[1 - e^{-\lambda(\Delta + \delta)}\right]}$$

and taking $\bar{\Delta} = \Delta + \delta$, we obtain for W the expression:

$$W = \frac{\left[1 - e^{-\lambda\Delta}\right]\left[1 - e^{-\lambda T}\right]}{\left[1 - e^{-\lambda\bar{\Delta}}\right]} \qquad (29)$$

Using (15) and substituting w_v and W by their values obtained from (28), (29) and the determination of λ through $\frac{d}{d\lambda} \ell n\, P = 0$, we obtain:

$$s - \frac{\Delta}{1 - e^{-\lambda\Delta}} + \frac{\Delta}{e^{\lambda\Delta} - 1} + \frac{T}{e^{\lambda T} - 1} - \frac{\bar{\Delta}}{e^{\lambda\bar{\Delta}} - 1} = 0 \qquad (30)$$

where

$$s = \frac{\sum_{v=1}^{k} n_v t_v}{m} \qquad (31)$$

Since:

$$- \frac{\Delta}{1 - e^{-\lambda\Delta}} + \frac{\Delta}{e^{\lambda\Delta} - 1} = - \Delta \qquad (32)$$

we obtain the expression:

$$s - \Delta + \frac{T}{e^{T\lambda} - 1} + \frac{\bar{\Delta}}{1 - e^{\lambda\bar{\Delta}}} = 0 \qquad (33)$$

It is easy to check that

$$- \Delta + \frac{\bar{\Delta}}{1 - e^{\lambda\bar{\Delta}}} = \delta - \frac{\bar{\Delta}}{1 - e^{-\lambda\bar{\Delta}}} \qquad (34)$$

giving the following final expression for the calculation of λ or τ for non consecutive intervals:

$$s + \delta - \frac{\bar{\Delta}}{1 - e^{-\bar{\Delta}/\tau}} + \frac{T}{e^{T/\tau} - 1} = 0 \qquad (35)$$

Taking $\bar{\Delta} = \Delta$ or $\delta = 0$, we have the case of adjacent intervals and (35) reduce to (15).

Determination of Standard Deviation.

Peierls[1] shows that the variance S is given by:

$$S = \frac{1}{m} \left(\left| \frac{f(\tau)}{f''(\tau)} \right| \right) \qquad (36)$$

when f (τ) is defined by:

$$P = \text{const.} \left[f(\tau) \right]^m$$

and the relation f/f" must be calculated at the τ value that makes P maximum.

The f (τ) values for adjacent and non adjacent intervals are:

$$f(\tau) = \frac{e^{-s/\tau}}{1 - e^{-T/\tau}} \left(e^{\Delta/\tau} - 1 \right) \quad \text{(adjacent intervals)} \qquad (38)$$

$$f(\tau) = \frac{\left[e^{-\bar{\Delta}/\tau} - 1 \right] e^{-s/\tau}}{1 - e^{-T/\tau}} \cdot e^{-\delta/\tau} \quad \text{(non-adjacent intervals)} \qquad (39)$$

Perfoming the calculation of the f/f", the following expression for the variances are obtained:

$$S = \frac{1}{m\tau^{4}} \left\{ \frac{\Delta}{1 - e^{-\Delta/\tau}} \left[(s - T) + (s - \Delta) \right] - s\,(s - T) \right\}^{-1} \quad (40)$$

(adjacent intervals)

$$S = \frac{1}{m\tau^{4}} \left\{ \frac{\bar{\Delta}}{1 - e^{-\bar{\Delta}/\tau}} \left[(s - T) + (s - \Delta) \right] - s\,(s - T) \right\}^{-1} \quad (41)$$

(non-adjacent intervals)

<u>Application of K. Pearson's moments method.</u>

Let us take:

$$s = \frac{\sum_{v=1}^{k} n_v t_v}{m} = \sum_{v=1}^{k} \frac{w_v}{w}\, t_v \quad (42)$$

where w_v = probability that a particle falls in the v^{th} interval;

w = probability that a particle falls in the interval from zero to T.

Introducing the values already obtained for w_v and w, in the case of adjacent intervals, i.e.;

$$w_v = (e^{\lambda\Delta} - 1)\; e^{-\lambda\Delta v} \quad (43)$$

$$w = (1 - e^{-\lambda T}) \quad (44)$$

we will have, noting that $t_v = v\Delta$

$$s = \frac{e^{\lambda\Delta} - 1}{1 - e^{-\lambda T}} \cdot \sum_{v=1}^{k} e^{-\lambda\Delta v}\, v\Delta \qquad (45)$$

Hence,

$$s - \frac{\Delta}{1 - e^{-\lambda\Delta}} + \frac{T}{e^{\lambda T} - 1} = 0 \qquad (46)$$

which is the same expression as obtained in (18). Using the expression w_v and w for non adjacent intervals, we would obtain (33).

Mean life determination: Long mean life case.

If the mean life to be determined is long compared with the intervals of counting, the general expression (35) for non adjacent intervals can be simplified.

Indeed, if we can take:

$$\bar{\Delta} = \Delta + \delta \simeq \delta \qquad (47)$$

the expression (35) reduces to:

$$s + \frac{T}{e^{T/\tau} - 1} - \frac{\delta}{e^{\delta/\tau} - 1} = 0 \qquad (48)$$

The condition (47) is experimentally well checked in several experiments measuring mean life greater than a few hours.

To analyse the error involved in the use of (48) it is enough to note that (48) is derived from (35), using:

$$\frac{\bar{\Delta}}{1-e^{-\bar{\Delta}/\tau}} \simeq \frac{-\delta}{e^{-\delta/\tau}-1} \qquad (49)$$

But, to first order in (Δ/δ), we have:

$$\frac{\bar{\Delta}}{1-e^{-\bar{\Delta}/\tau}} \simeq \frac{\delta}{1-e^{-\delta/\tau}}\left\{1+\frac{\Delta}{\delta}\left(1-\frac{\frac{\delta}{\tau}e^{-\delta/\tau}}{1-e^{-\delta/\tau}}\right)\right\} \qquad (50)$$

In this way, the error involved using (48) is always less than (Δ/δ); and if $\delta/\tau \ll 1$, such an error approaches (Δ/τ). Even if $\delta/\tau = 0.2$ and $\frac{\Delta}{\delta} = 0.1$, the error in (48) is less than 1%.

Bibliography.

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Appendix.

To exemplify the method of calculation introduced in this paper, we will consider a set of results obtained in an experiment with a pulsed neutron source, and performed to measure the mean life of thermal neutrons in water. (Table I).

1. Adjacent Intervals: $\Delta = 10\ \mu\text{sec}$

Data:

$$\Delta = 10\ \mu\text{sec}$$
$$T = 200\ \mu\text{sec}$$
$$\delta = 0$$
$$t_v = 10,\ 20,\ 30\ \ldots\ 200\ \mu\text{sec}$$
$$\sum n_v t_v = 2{,}251{,}700 \qquad m = 36{,}530$$

$$s = \frac{1}{m}\sum n_v t_v = 61.64\ \therefore\ \frac{s}{\Delta} = 6.164$$

Rearranging the expression (20), we will obtain:

$$\frac{s}{\Delta} - \frac{1}{1 - e^{-\Delta/\tau}} + \frac{T/\Delta}{e^{\left(\frac{T}{\Delta}\right)\cdot\left(\frac{\Delta}{\tau}\right)} - 1} = 0 \qquad (20')$$

Given reasonable values for Δ/τ, we can construct the table II, with $T/\Delta = 20$.

From this table, we obtain:

For: $S/\Delta = 6.164$

$\tau = 67\ \mu\text{sec}$

Since, in this case, $\Delta \ll \tau$, we could apply the approximate formula (23) of Peierls; the result would be:

$$\tau = 67\ \mu\text{sec}$$

2. <u>Adjacent Intervals</u>: $\Delta = 30\ \mu\text{sec}$

If $\Delta = 30\ \mu\text{sec}$

$T = 180\ \mu\text{sec}$

$t_v = 30,\ 60\ \ldots\ 180\ \mu\text{sec}$

$\sum n_v t_v = 2{,}502{,}600$; $m = 35{,}820$

$s = 69.866$ $\therefore$ $\tau = 67\ \mu\text{sec}$

The application of the approximate formula of Peierls would give, in this case:

$$\tau = 69{,}5\ \mu\text{sec}$$

with a discrepance of about 3,7%.

3. <u>Non-adjacent Intervals</u>: $\bar{\Delta} = 20\ \mu\text{sec}$

If $\bar{\Delta} = 20\ \mu\text{sec}$

$\delta = 10\ \mu\text{sec}$

$T = 200\ \mu\text{sec}$

$t_v = 10,\ 30,\ 50\ \ldots\ 190\ \mu\text{sec}$

$\sum n_v t_v = 1{,}112{,}500$ $m = 19{,}310$

$s = 57.613$

After rearranging equation (35) can be written as:

$$\frac{s}{\bar{\Delta}} + \frac{\delta}{\bar{\Delta}} - \frac{1}{1 - e^{-\bar{\Delta}/\tau}} + \frac{\left(\frac{T}{\bar{\Delta}}\right)\bar{\Delta}}{e^{\frac{T}{\bar{\Delta}}(\bar{\Delta}/\tau)} - 1} = 0 \qquad (35')$$

and constructing the table III, for $T/\bar{\Delta} = 10$ we obtain for

$$\bar{\Delta}/\tau = 0.294$$
$$\tau = 68\ \mu\text{sec}$$

4. Regression analysis.

For comparison, the data in the table I were adjusted by means of a regression analysis and the value of τ obtained was:

$$\tau = 68\ \mu\text{sec}\ .$$

in very good agreement with the values calculated using the exact expression deduced in this paper.

TABLE I

t_v μsec	n	t_v μsec	n
50	4950	150	1210
60	4780	160	990
70	4040	170	900
80	3500	180	710
90	3040	190	670
100	2610	200	550
110	2110	210	480
120	1890	220	470
130	1530	230	380
140	1390	240	330

TABLE II

Δ μsec	Δ/τ	$\frac{1}{1 - e^{-\Delta/\tau}}$	$\frac{20}{e^{20\,\Delta/\tau} - 1}$	s/Δ
10	0.125	8.511	1.789	6.722
10	0.143	7.507	1.215	6.292
10	0.166	6.514	0.741	5.773

TABLE III

$\bar{\Delta}/\tau$	$\frac{1}{1 - e^{-\bar{\Delta}/\tau}}$	$\frac{20}{e^{20\bar{\Delta}/\tau} - 1}$	$\frac{s + \delta}{\Delta}$
0.10	10.504	5.820	4.684
0.15	7.178	2.871	4.307
0.20	5.515	1.565	3.950
0.25	4.520	0.894	3.626
0.30	3.858	0.524	3.334
0.35	3.386	0.311	3.075