

better flux for cell homogenization and condensation (normally 25 → 6 groups) than the traditionally used asymptotic flux obtained by assuming that each pin-cell type has zero net in-current. At the same time, the present method reproduces this asymptotic flux in the limit of an infinite uniform lattice. Moreover, the net currents it yields are utilized in determining auxiliary sources for improving cell homogenization.³

1. I. CARLVIK, "A Method for Calculating Collision Probabilities in General Cylindrical Geometry and Applications to Flux Distributions and Dancoff Factors," *Proc. Int. Conf. Geneva* (1964), 2, 225 (1965).
2. ZBIGNIEW WEISS, "Nodal Equations Derived from Invariant Imbedding Theory," *Nucl. Sci. Eng.*, **48**, 235 (1972).
3. RUDI STAMM'LER, "A Note on Pin-Cell Homogenization," *Trans. Am. Nucl. Soc.*, **26**, 222 (1977).

4. On Multiregion Problems in Plane Geometry, Yuji Ishiguro (IEA-Brazil)

A general systematic method to reduce multiregion problems in plane geometry to regular integral equations for the coefficients of singular-eigenfunction expansions is proposed. The method requires only half-range orthogonality relations and is applicable in two-group as well as one-group theory.

Since the publication of Case's fundamental paper, the transport equation for plane geometry has been studied extensively in various models. General solutions have been obtained and their full-range and half-range completeness properties have been proved in several models. However, though various orthogonality relations of the eigenfunctions have been obtained to solve ensuing equations for the expansion coefficients, types of solvable problems have been quite limited due to difficulties in removing the singularities.

Our purpose here is to propose a general method to derive a set of regular integral equations for the coefficients from the set of singular integral equations that results from boundary and interface conditions. Since we demonstrate the method in the accompanying paper¹ in a specific problem, here we summarize in a general way the steps involved:

1. At an interface, separate the continuity condition into two equations, one for $\mu \in (0,1)$, the other for $\mu \in (-1,0)$.
2. (a) To the $\mu \in (0,1)$ equation apply half-range orthogonality relations for the right-side medium.
(b) In the $\mu \in (-1,0)$ equation change μ to $-\mu$ and then apply half-range orthogonality relations for the left-side medium.
3. (a) If any singularity remains in step 2(a), consider the interface (or boundary) condition for $\mu > 0$ at the left-side boundary of the left-side medium and generate the same singularity, subtract the result from the equation in step 2(a) and remove the singularity.
(b) For step 2(b) consider the right-side interface of the right-side medium (consider $\mu < 0$).
4. If singularities remain in step 3, repeat the process, generating the same singularities at different interfaces.

We have applied the method to several problems in both the one-group and two-group models of neutron transport theory for isotropic scattering. All singularities can be removed in the form

$$\frac{\exp(a/\nu) - \exp(a/\mu)}{\nu - \mu};$$

thus, iterative solutions of the expansion coefficients can be obtained by the standard method.

One disadvantage of the method is that it does not solve two-half-space problems analytically in one-group theory, as can be done by the use of two-media orthogonality relations. One may find another in that some of the regularized equations become quite long, involving all unknown coefficients, as the number of regions increases. However, in the few problems for which numerical works have been completed, including a three-region problem in one-group theory, we have not encountered any difficulty; convergence is smooth and a high degree of accuracy can be achieved.

As we show in the accompanying paper,¹ the method can be used in two-group theory and we believe other two-group problems involving dissimilar media; e.g., the critical problem for reflected slab reactors and the transmission of neutrons through two slabs can now be solved. However, there is one model problem that we have not been able to regularize—the cell problem. This is due to the facts that the cell problem is geometrically an infinite array and that the processes of regularization terminate at an outermost boundary or interface, though in one-group theory they can be terminated at a boundary of symmetry.

Though the method proposed here has a few limitations and disadvantages, we believe it can be used to solve various problems that have long remained unsolvable.

1. Y. ISHIGURO, *Trans. Am. Nucl. Soc.*, **26**, 224 (1977).

5. Two-Media Milne Problem in Two-Group Neutron Transport Theory, Yuji Ishiguro (IEA-Brazil)

The Milne problem for a half-space bounded by a slab of dissimilar medium is solved numerically for the first time in the two-group isotropic scattering model.

In spite of the well-known merits of the two-group model in problems involving dissimilar media and extensive investigations in recent years, two-media problems have remained unsolved in exact transport theory. The difficulty was that no method was known to remove all singularities in the integral equations for the expansion coefficients. Recently, Ishiguro and Maiorino¹ circumvented this difficulty by combining the Case method with the principle of invariance and reported several numerical results. However, their technique has been successful only for the case of two half-spaces. Here we

TABLE I
Extrapolated Endpoint z_0

| α | z_0 |
|-------------------|---------|
| 0 (Refs. 3 and 5) | 0.66583 |
| 1 | 1.6228 |
| 2 | 2.4628 |
| 5 | 4.0538 |
| 10 | 4.7189 |
| ∞ (Ref. 1) | 4.7944 |

consider a problem involving a finite medium, the Milne problem in a half-space ($x \geq 0$) of medium 2 bounded by a slab of thickness α of medium 1 ($-\alpha \leq x \leq 0$), and show that all singularities can be removed by the method proposed in the accompanying paper.²

The boundary condition at $x = -\alpha$ can be written as

$$A(\nu_1)\Phi_1(\nu_1, \mu) + \int_0^1 \Phi_1(\nu, \mu)A(\nu)d\nu = -A(-\nu_1)\Phi_1(-\nu_1, \mu)E(\nu_1) - \int_0^1 \Phi_1(-\nu, \mu)A(-\nu)E(\nu)d\nu, \quad \mu \in (0, 1), \quad (1)$$

and we write the continuity condition at $x = 0$ in two equations

$$A(-\nu_1)\Phi_1(\nu_1, \mu) + \int_0^1 \Phi_1(\nu, \mu)A(-\nu)d\nu = G\Phi_2(\eta_1, \mu) + B(\eta_1)G\Phi_2(-\eta_1, \mu) + G\int_0^1 \Phi_2(-\eta, \mu)B(\eta)d\eta - A(\nu_1)\Phi_1(-\nu_1, \mu)E(\nu_1) - \int_0^1 \Phi_1(-\nu, \mu)A(\nu)E(\nu)d\nu, \quad \mu \in (0, 1), \quad (2)$$

$$B(\eta_1)\Phi_2(\eta_1, \mu) + \int_0^1 \Phi_2(\eta, \mu)B(\eta)d\eta = -\Phi_2(-\eta_1, \mu) + A(\nu_1)G^{-1}\Phi_1(\nu_1, \mu)E(\nu_1) + A(-\nu_1)G^{-1}\Phi_1(-\nu_1, \mu) + G^{-1}\int_0^1 \Phi_1(\nu, \mu)A(\nu)E(\nu)d\nu + G^{-1}\int_0^1 \Phi_1(-\nu, \mu)A(-\nu)d\nu, \quad \mu \in (0, 1), \quad (3)$$

where $\Phi(\xi, \mu)$, $\xi \in (-1, 1)$, is a 2×2 matrix, $A(\pm\nu)$ and $B(\eta)$ are two-vectors, $E(\xi) = \exp(-\alpha/\xi)$, and G is a constant 2×2 diagonal matrix which comes in from symmetrizing the scattering matrices. Further, we have assumed one pair of discrete eigenvalues for both media and similarly ordered groups.

Applying the half-range orthogonality relations³ of medium 1 eigenfunctions to Eqs. (1) and (2), we can isolate the coefficients in the left-side expansions:

$$A(\nu_1) = -\nu_1 N_1^{-1}(\nu_1)U_1(\nu_1)C_1 H_1^{-1}(\nu_1)C_1^{-1}Y_1(\nu_1) \quad (4a)$$

$$A(\nu) = -\nu N_1^{-1}(\nu)W_1(\nu)C_1 H_1^{-1}(\nu)C_1^{-1}Y_1(\nu) \quad (4b)$$

$$A(-\nu_1) = \nu_1 N_1^{-1}(\nu_1)U_1(\nu_1)C_1 H_1^{-1}(\nu_1)C_1^{-1}Y_2(\nu_1) + A_0(-\nu_1) \quad (5a)$$

$$A(-\nu) = \nu N_1^{-1}(\nu)W_1(\nu)C_1 H_1^{-1}(\nu)C_1^{-1}Y_2(\nu) + A_0(-\nu), \quad (5b)$$

where two-vectors $Y_1(\xi)$ and $Y_2(\xi)$ are regular functionals involving expansion coefficients, the second terms in Eqs. (5) are independent of the coefficients, and other functions are the same as in previous work.^{3,4}

If we next apply the half-range orthogonality relations for medium 2 to Eq. (3), we can isolate the coefficients in the left side. However, the continuum coefficient $B(\eta)$ involves singular integrals of $A(\nu)$, and this singularity is the reason why this problem has not been solved. Now, if we multiply Eq. (1) by

$$\mu \Phi_2(\eta, \mu)G^{-1} \begin{bmatrix} E(\sigma_2 \eta / \sigma_1) & 0 \\ 0 & E(\eta) \end{bmatrix}$$

and integrate over $\mu \in (0, 1)$, we obtain in the left side the same singular integrals as in the equation for $B(\eta)$, except for the E functions. Then, subtracting between the two equations, we obtain

$$B(\eta) = \eta_1 N_2^{-1}(\eta_1)U_2(\eta_1)C_2 H_2^{-1}(\eta_1)C_2^{-1} [Y_3(\eta_1) + Y_4(\eta_1)] + B_0(\eta_1) \quad (6a)$$

$$B(\eta) = \eta N_2^{-1}(\eta)W_2(\eta)C_2 H_2^{-1}(\eta)C_2^{-1} [Y_3(\eta) + Y_4(\eta)] + B_0(\eta), \quad (6b)$$

where $Y_3(\xi)$ is a regular functional and $Y_4(\xi)$ is a functional involving integrals with removable singularities.

We have solved Eqs. (4), (5), and (6) iteratively for the same data sets as in previous works^{1,3,5}; medium 2 is pure light water and medium 1 is borated water at 1.025 b/H. We report for several values of α the extrapolated endpoint defined by the zero-extrapolation of the asymptotic total flux in medium 2.

1. Y. ISHIGURO and J. R. MAIORINO, *Nucl. Sci. Eng.*, in print.
2. Y. ISHIGURO, *Trans. Am. Nucl. Soc.*, **26**, 224 (1977).
3. C. E. SIEWERT and Y. ISHIGURO, *J. Nucl. Energy*, **26**; 251 (1972).
4. J. T. KRIESE, C. E. SIEWERT, and Y. YENER, *Nucl. Sci. Eng.*, **50**; 3 (1973).
5. D. R. METCALF and P. F. ZWEIFEL, *Nucl. Sci. Eng.*, **33**, 318 (1968).

6. Conjugate Gradient Solutions to the Discrete Ordinates Equations, E. E. Lewis* (LASL)

In an attempt to provide more effective solution algorithms for S_N equations,¹ the use of generalized conjugate gradient methods² has been examined.³ In these methods, the solution of a symmetric matrix equation $Ax = b$ is obtained by first splitting the coefficient matrix, $A = M - N$, and then forming a series of residual vectors r_i such that $r_i^T M r_j = 0$, $i \neq j$. Like the original conjugate gradient method,⁴ for which $M + I$, an exact solution will be obtained when the iteration number is equal to the dimension of x . While requiring that M^{-1} be performed at each iteration, generalized conjugate gradient methods have been shown often to result in greatly increased rates of convergence.² To apply such methods to the S_N equations, the equations must be cast in a symmetric form, and the S_N sweeping of the space-angle grid must be retained to obviate storage of the coefficient matrix elements.

The diamond-differenced S_N equations may be written as

$$L_\ell \psi_\ell = \frac{1}{4\pi} \sigma_S \sum_{\ell'} \omega_{\ell'} \psi_{\ell'} + \frac{1}{4\pi} q, \quad \ell = 1, 2, \dots, L,$$

where ψ_ℓ is the vector of $\psi(r, \Omega_\ell)$ at the spatial mesh points, the streaming-collision matrix L_ℓ is lower triangular, σ_S is diagonal, and ω_ℓ are the quadrature weights. An equivalent partitioned matrix equation is

$$L\psi = W\psi + Q, \quad (1)$$

where ψ is a vector made up of ψ_1, \dots, ψ_L . The standard solution algorithm, iteration on the scattering source, corresponds to

$$\Phi = K(\sigma_S \Phi + q) \quad K \equiv \frac{1}{4\pi} \sum_{\ell} \omega_{\ell} L_{\ell}^{-1}, \quad (2)$$

where Φ is the scalar flux vector, and L_{ℓ}^{-1} corresponds to sweeping the mesh in direction Ω_ℓ .

To obtain conjugate gradient methods, Eqs. (1) and (2) are symmetrized as follows:

$$(L^T - W^T)(L - W)\psi = (L^T - W^T)Q \quad (3)$$

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