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# The spin-one Duffin-Kemmer-Petiau equation revisited: analytical study of its structure and a careful choice of interaction

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## PAPER

## The spin-one Duffin-Kemmer-Petiau equation revisited: analytical study of its structure and a careful choice of interaction

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E-mail: [marzieh.baradaran@uhk.cz](mailto:marzieh.baradaran@uhk.cz), [luismiguel.nieto.calzada@uva.es](mailto:luismiguel.nieto.calzada@uva.es), [oliveira.phys@gmail.com](mailto:oliveira.phys@gmail.com) and [saber.zarrinkamar@uva.es](mailto:saber.zarrinkamar@uva.es)**Keywords:** spin-one duffin-kemmer-petiau equation, ab-normal parity-state, lie algebra, Heun functions**Abstract**

The Duffin-Kemmer-Petiau equation is investigated for spin one bosons with the so-called natural (normal) and unnatural (abnormal) parity states for non-minimal vector interactions. To illustrate the current state of knowledge about the equation, a thorough but concise discussion is made on what can be achieved analytically within this framework for well-known phenomenological interactions, including Coulomb, soft-core, Cornell, Kratzer, and exponential type interactions. In the non-exponential cases, the equation, depending on the chosen interaction, is studied in relation to the confluent, doubly-confluent, and biconfluent Heun functions. Furthermore, to show the need for careful treatment of various parity states, a Kratzer-type potential, such as a generalized Coulomb interaction, is discussed in depth using the Lie algebraic approach, showing the need for careful analysis of abnormal parity states in a fairly explicit way. The energies obtained are discussed using some figures to explicitly show the different regimes, as well as the absence of the Klein paradox. Finally, some directions for future work that would undoubtedly need to be explored in this field are discussed.

**1. Introduction**

In relativistic quantum mechanics, the Klein-Gordon, Dirac, and Proca-Weinberg equations describe, respectively, spin-zero, spin-1/2, and spin-one particles, and the Duffin-Kemmer-Petiau (DKP) equation provides a framework within which both spin-zero and spin-one particles can be investigated [1–4]. The case of spin zero, due to a structure similar to the Klein-Gordon equation, has been frequently studied in the literature [4, 5]. The spin-one version of the equation becomes more interesting when we remember the complicated nature of the Proca and Weinberg equations, as well as the scant literature that exists on them. On the other hand, the DKP equation, due to the way it is represented, which appears in the form of a ten-spinor wave function, is a rich basis for analyzing various interactions and fields. The structure of this equation has motivated recent papers investigating the problem with a variety of physical interactions and concepts: for example, Nedjadi and Barrett [6] considered the free particle, potential well, and Coulomb problem for vector bosons, solutions for DKP with a linear interaction were found in [7], and the DKP oscillator was analyzed within a noncommutative approach in [8]. The similarity of the structures in DKP and Dirac equations was discussed in the conceptual work of Okniński [9] and the creation rate within the de Sitter framework was obtained using the Bogoliubov transformation in [10]. Castro and de Oliveira provided a review of the equation with non-minimal vector interactions, including the linear one [11], and the Galilean DKP theory was formulated at finite temperature to study Bose-Einstein condensation [12]. An instructive survey on the equivalence of the DKP equation with the Klein-Gordon and Proca equations is provided in [13]. In [14] ground state solutions of a class

of generalized Coulomb interactions were derived using the Ansatz approach. The fractional Hamiltonian formulation of the equation in the Lagrangian formalism was restudied by Bouzid and Merad [15]. The equation in (1+1)-dimensions was analyzed in [16]. The associated Aharonov-Bohm problem was discussed in detail by Castro and Silva [17], and the field-theoretical analysis of the equation was formulated in [18], a work in which the corresponding Stefan-Boltzmann law was also discussed, as well as the Casimir effect at finite temperature. Later, de Montigny and Santos [19] generalized the formulation of the equation to arbitrary dimensions.

Solutions of the DKP equation with a Kratzer potential in a noncommutative approach were obtained in [20], a work in which some delicate points should have been carefully considered, as we will see in some of the following sections. Let us also mention that the non-relativistic limit of the equation was found in a non-commutative framework using the Foldy-Wouthuysen transformation and the Moyal-Weyl product [21]. The Dirac delta problem in the  $q$ -deformed formulation of the equation was solved by Sobhani *et al* [22], and using Dirac derivatives, Chargui and Cherif found the DKP equation valid within a  $\kappa$ -Minkowski spacetime [23]. Increasing the spatial dimensions, the (2 + 1)-dimensional DKP oscillator influenced by an external magnetic field and the corresponding canonical ensemble thermodynamic properties were reviewed in [24]. On the other hand, the study of the momentum space of the three-dimensional DKP oscillator within the Snyder-de Sitter framework was performed in [25]. In a conceptual paper, Chragui and Dhahbi analyzed an extended DKP oscillator for spin-one bosons, exhaustively studying various parity states [26], and subsequently treated the DKP oscillator in three spatial dimensions in a noncommutative plane, showing the salient related physical aspects [27].

In recent years there has been intense activity in this area, with the following results being noteworthy: the rotational symmetries of the equation within the (1 + 2)-dimensional Gürses spacetime background were commented on by Candemir and Ahmed [28], the equation with a hyperbolic tangent potential was investigated by Valladares and Rojas [29], the DKP oscillator problem in one and two spatial dimensions and the associated thermal properties were analyzed within a noncommutative framework [30], a new non-minimal coupling associated with a  $q$ -deformed approach was considered within the framework of the equation and the thermodynamic properties of the  $q$ -deformed oscillator were studied [31], and Chargui *et al* generalized the DKP oscillator in a form that includes an additional spin-orbit term [32], to mention only the most prominent ones. The motivation for such studies is not exceptional because spin-one systems are present in many areas, including molecular systems [33, 34], isotropic spin-1 antiferromagnetic systems [35], Cooper pairs [36], charged impurities [37], etc. Very recently, Charqui and Dhahbi [38] have addressed a very interesting and applied aspect of the DKP equation: they have connected the equation in two spatial dimensions in a kind of atom-field interaction similar to the Jaynes-Cummings model, which may be a great step in the field if we remember the limitations of other models. However, this idea is quite recent and absolutely open to debate, with many questions and challenges that need to be properly answered. It is worth mentioning that the problem of vector bosons, photons, has also been discussed in several articles in different frameworks [39–43]

The present work focuses on two main points: (i) a general analysis of the structure of the spin one DKP equation and what can be derived analytically from a fairly general class of phenomenological interactions and (ii) a clear and detailed discussion of so-called unnatural or abnormal parity states, which have not been investigated before in detail. To this end, we start in section 2 from the ordinary differential equations governing the DKP spinor components to avoid extensive text citations of previous work. The reader must either be familiar with the spin one DKP equation, or carefully read the original works on the derivation of various parity states [6, 7, 11, 26] in order to understand the concepts and calculations. We next present the Lie-algebraic approach to tackle the equations arising in the case of a carefully chosen Kratzer-type interaction. We derive the general solution in section 3 and analyze the energies for both the natural and unnatural parts to provide a clear discussion on the physics of the equation. In section 4, without going into details, the forms of the equation with a large class of phenomenological interactions are discussed in terms of Heun functions and other approaches in mathematical physics. Concluding remarks are included in the last section.

## 2. The structure of DKP equation

Before starting with the essential mathematical developments, let us mention some points about the limitation in the analytical study of the spin one DKP equation. To avoid overlaps with previous publications, we will make some statements and refer to the original articles [6, 7, 11] to support them. A look at equations (61) and (62) of [6] reveals that the general form of the equation appears as a system of coupled second-order differential equations that, to the best of our knowledge, has not been solved analytically. One might think that the homogeneous part of equation (61) in [6] is the general form of the Heun equation [44] and that, in this case, the solution can be reported. However, this is not true since the spinor components that appear there are coupled and we cannot simply think of a single dependent variable problem. This very complicated form of the equation

has forced the authors to focus on the case of  $J = 0$  in order to report the analytical solutions or, as will be seen soon, to consider a zero term in part of the interaction [6, 7, 11].

The components of the spin one DKP equation are denoted as the parity states  $(-1)^j$  and  $(-1)^{j+1}$ . This terminology originates from the behavior of the associated spherical harmonics of each spinor under Maxwell-like equations. Those that retain their value of  $J$  are called  $(-1)^j$  components, or natural/normal/magnetic-like parity states, and those with a one-unit increase or decrease in the value of their  $J$  are called  $(-1)^{j+1}$  components, or unnatural/abnormal-electric-like parity states. The connection to electric and magnetic fields comes simply from the behavior of these fields under parity, as they have negative and positive parities, respectively. Let us now review the form that the final equations take and emphasize that the reader who is not an expert on the subject should first read the articles [6, 7, 11, 26].

**2.1.  $(-1)^j$ -parity components**

Obviously it is not appropriate to present here the entire content of [6, 7, 11], but we do believe it is essential to mention the most relevant aspects of the ten-component structure of the problem. After various calculations, it can be shown that the original DKP equation can be expressed in terms of three auxiliary components, one of which,  $F_0(r)$ , represents the so-called normal (natural) parity states and the other two, i.e.  $H_0(r)$  and  $\phi(r)$ , govern the abnormal (unnatural) parity states, and common solutions ( $E$ ) must be sought in the equations governing  $H_0(r)$  and  $\phi(r)$ . Therefore, although these three second-order ordinary differential equations produce the spectrum of the system for different parity states, their eigenfunctions are not those that originally appear in the DKP spinor. However, the spinor components can be found since the relations are given completely in [6, 7, 11].

As already mentioned above, we work on a ten-component basis, in which, roughly speaking, we denote by  $F_0(r)$  the term related to the original component of parity preserved through some relations. Once  $F_0(r)$  is determined, all data related to natural parity states are derived. After some algebra (not included here to avoid repetition), it follows that this function must satisfy the differential equation [6, 7, 11]

$$\frac{d^2F_0(r)}{dr^2} + \left[ E^2 - M^2 - \frac{dA_r}{dr} - \frac{j(j+1)}{r^2} + A_0^2 - A_r^2 \right] F_0(r) = 0, \tag{1}$$

where  $j$  denotes the order of the associated spherical harmonics. The term  $A_r(r)$  is a spherically symmetric potential and the scalar part of the original DKP equation. We consider  $A_r(r)$  and  $A_0(r)$  in the form of the modified Kratzer potential  $V_{MK} = D_e \left( \frac{r-r_e}{r} \right)^2$ , with  $D_e > 0$  [45], indeed

$$A(r) = \frac{x_r}{r} + \frac{y_r}{r^2} + z_r, \quad A_0(r) = \frac{x_0}{r} + \frac{y_0}{r^2} + z_0, \tag{2}$$

In this form,  $x_r < 0, y_r > 0$  and  $z_r > 0$ . Substitution of the potential (2) in (1) yields

$$\left\{ \frac{d^2}{dr^2} + E^2 - M^2 + z_0^2 - z_r^2 + \frac{2(x_0z_0 - x_rz_r)}{r} + \frac{2(y_0z_0 - y_rz_r) - j(j+1) + x_r + x_0^2 - x_r^2}{r^2} + \frac{2(x_0y_0 - x_r y_r + y_r)}{r^3} + \frac{y_0^2 - y_r^2}{r^4} \right\} F_0(r) = 0. \tag{3}$$

Let us now review the equations governing the unnatural parity states.

**2.2.  $(-1)^{j+1}$ -parity components**

In this case, in order to report analytical solutions, we have to consider  $A_0 = 0$ . In this case, we have to consider two equations for the spinor components [6, 7, 11]:

$$\frac{d^2H_0}{dr^2} + \left[ E^2 - M^2 + \frac{dA_r}{dr} - \frac{j(j+1)}{r^2} - A_r^2 \right] H_0 = 0 \tag{4}$$

and

$$\frac{d^2\phi}{dr^2} + \left[ E^2 - M^2 - \frac{dA_r}{dr} - \frac{j(j+1)}{r^2} - \frac{A_r}{r} - A_r^2 \right] \phi = 0, \tag{5}$$

which take the following forms for the potential (2):

$$\left\{ \frac{d^2}{dr^2} + E^2 - M^2 - z_r^2 - \frac{2x_rz_r}{r} - \frac{j(j+1) + x_r(x_r+1) + 2y_rz_r}{r^2} - \frac{2(x_r+1)y_r}{r^3} - \frac{y_r^2}{r^4} \right\} H_0(r) = 0, \tag{6}$$

and

$$\left\{ \frac{d^2}{dr^2} + E^2 - M^2 - z_r^2 - \frac{(2x_r + 1)z_r}{r} - \frac{j(j+1) + x_r^2 + 2y_r z_r}{r^2} + \frac{(1 - 2x_r)y_r}{r^3} - \frac{y_r^2}{r^4} \right\} \phi(r) = 0. \quad (7)$$

In the next section we will look for analytical solutions to the three equations (3), (6) and (7).

### 3. Quasi-polynomial solutions

First we review the basic concepts of quasi-exact solubility [46] and then we will comment on the solutions and the spectrum associated with each of the three differential equations obtained above.

#### 3.1. Algebraic structure with Kratzer-like interaction

It is clear that the three equations (3), (6) and (7) appear in the form of Schrödinger equations  $H\psi(r) = 0$  with Hamiltonians of the same type

$$H = \frac{d^2}{dr^2} + \alpha + \frac{\beta}{r} + \frac{\omega}{r^2} + \frac{\delta}{r^3} + \frac{\sigma}{r^4}, \quad (8)$$

already analyzed in [46–48]. Applying the transformation

$$\psi(r) = r^{1-\eta} \exp \left[ -\sqrt{-\alpha} r - \frac{\sqrt{-\sigma}}{r} \right] R(r), \quad \eta = \frac{\delta}{2\sqrt{-\sigma}}, \quad (9)$$

in which  $1 - \eta > 0$  and  $\sigma, \alpha < 0$ , equation (8) can be written in the form  $\tilde{H}R(r) = 0$  with

$$\tilde{H} = r^2 \frac{d^2}{dr^2} - 2(\sqrt{-\alpha} r^2 - (1 - \eta)r - \sqrt{-\sigma}) \frac{d}{dr} - (ar + b), \quad (10)$$

where

$$a = 2(1 - \eta)\sqrt{-\alpha} - \beta, \quad b = \eta(1 - \eta) + 2\sqrt{\alpha\sigma} - \omega. \quad (11)$$

The differential operator  $\tilde{H}$  can be written as an element of the universal enveloping algebra of  $sl(2)$  as [46–50],

$$\tilde{H}_{QES} = -\mathcal{J}_n^+ \mathcal{J}_n^- + 2\sqrt{-\alpha} \mathcal{J}_n^+ - 2\sqrt{-\sigma} \mathcal{J}_n^- + (2\eta - n - 2) \left( \mathcal{J}_n^0 + \frac{n}{2} \right) + b, \quad (12)$$

provided that

$$a = -2n\sqrt{-\alpha}, \quad n = 0, 1, 2, \dots \quad (13)$$

In (12) the generators of the algebra  $sl(2)$  are [46]

$$\mathcal{J}_n^+ = r^2 \frac{d}{dr} - n r, \quad \mathcal{J}_n^0 = r \frac{d}{dr} - \frac{n}{2}, \quad \mathcal{J}_n^- = \frac{d}{dr}, \quad n = 0, 1, 2, \dots, \quad (14)$$

and satisfy the commutation relations [46]

$$[\mathcal{J}^+, \mathcal{J}^-] = -2\mathcal{J}^0, \quad [\mathcal{J}^\pm, \mathcal{J}^0] = \mp \mathcal{J}^\pm, \quad (15)$$

that leave invariant the  $(n+1)$ -dimensional linear space of finite order polynomials  $P_{n+1}(r) = \langle 1, r, r^2, \dots, r^n \rangle$ . As a result,  $H_{QES}$  preserves the finite-dimensional space of polynomials

$$R(r) = \sum_{m=0}^n c_m r^m, \quad (16)$$

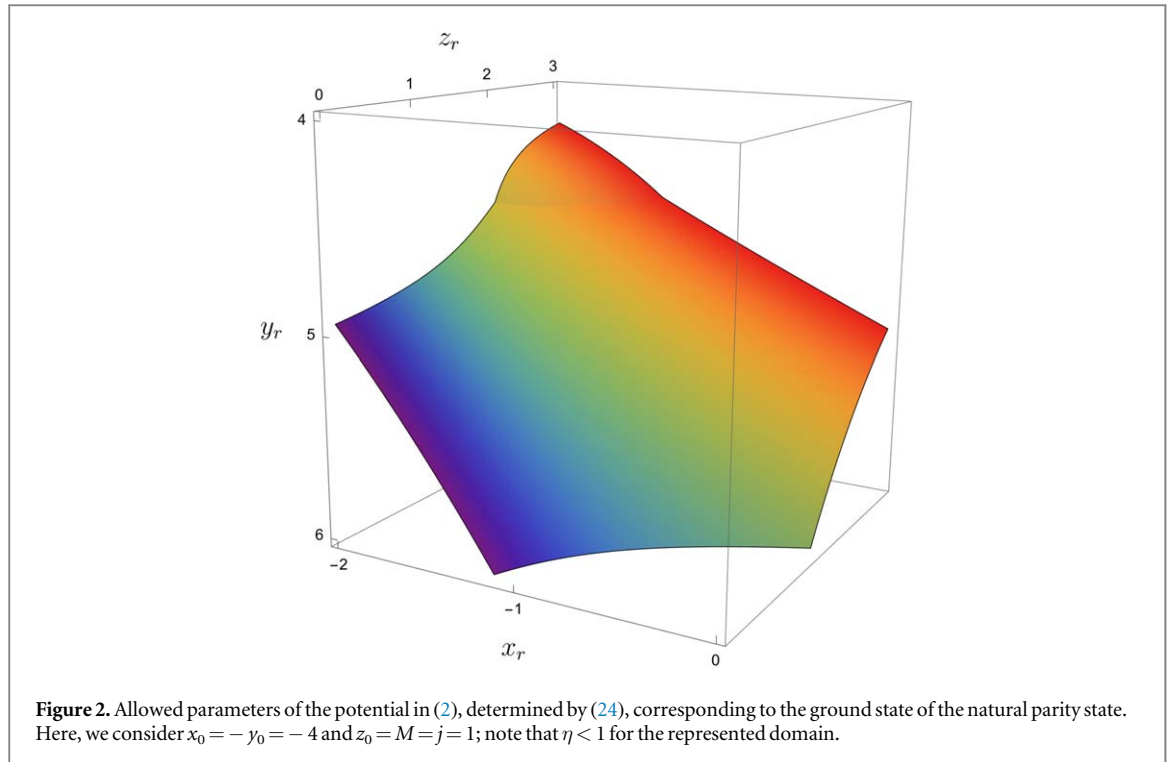
and we may finally write the wave-function (9) as [46]

$$\psi_n(r) = r^{1-\eta} \exp \left[ -\sqrt{-\alpha} r - \frac{\sqrt{-\sigma}}{r} \right] \sum_{m=0}^n c_m r^m, \quad (17)$$

where the coefficients  $c_m$  satisfy the three-term recursion relation

$$c_{m+1} = \frac{(b + m(2\eta - m - 1))c_m + 2(m - n - 1)\sqrt{-\alpha} c_{m-1}}{2(m+1)\sqrt{-\sigma}}, \quad \text{with} \quad c_{-1} = c_{n+1} = 0. \quad (18)$$





Regarding the relevant component in this case, from (17) we obtain that its expression is

$$F_{0,n}(r) = r^{1-(x_0y_0-x_r y_r+y_r)/\sqrt{y_r^2-y_0^2}} \exp \left[ -r\sqrt{M^2 - E_n^2 + z_r^2 - z_0^2} - \frac{\sqrt{y_r^2 - y_0^2}}{r} \right] \sum_{m=0}^n c_m r^m. \quad (22)$$

Note that as we mentioned above, we must have  $1 - \eta > 0$ , which specifically means that

$$\eta = \frac{x_0y_0 - x_r y_r + y_r}{\sqrt{y_r^2 - y_0^2}} < 1. \quad (23)$$

Moreover, note that the potential parameters in (21) and (22) must fulfill the constraint given by (19). For example, in the simplest case, corresponding to the ground state  $n = 0$ , as deduced from (19), the parameter restriction is given by  $b = 0$ . This condition, together with (11) and (20), explicitly leads to

$$\eta(1 - \eta) + 2\sqrt{(y_0^2 - y_r^2)(E_0^2 - M^2 + z_0^2 - z_r^2)} - 2(y_0z_0 - y_r z_r) + j(j + 1) + x_r^2 - x_r - x_0^2 = 0 \quad (24)$$

where  $\eta < 1$  and  $E_0$  are respectively given by (23) and (21). Figure 2 shows a graph of the allowed potential parameter in (2) based on the constraint (24).

Let us now consider the unnatural parity solutions.

### 3.3. $(-1)^{j+1}$ -parity states

As we have already seen, we have to consider two different types of spinor components: those of  $H_0(r)$  coming from (6) and those of  $\phi(r)$ , coming from (7). We first consider the equation governing  $H_0(r)$ .

#### 3.3.1. $H_0(r)$ solutions

To obtain the solutions for  $H_0(r)$  we have to compare (6) and (8), from which we deduce

$$\begin{aligned} \alpha &= E_n^2 - M^2 - z_r^2, & \beta &= -2x_r z_r, \\ \omega &= -(j(j + 1) + x_r(x_r + 1) + 2y_r z_r) \\ \delta &= -2(x_r + 1)y_r, & \sigma &= -y_r^2, \end{aligned} \quad (25)$$

from which, together with (13), the possible values of the solutions are deduced:

$$E_n^2 = -\left(\frac{x_r z_r}{x_r + n + 2}\right)^2 + M^2 + z_r^2, \quad n = 0, 1, 2, \dots \quad (26)$$

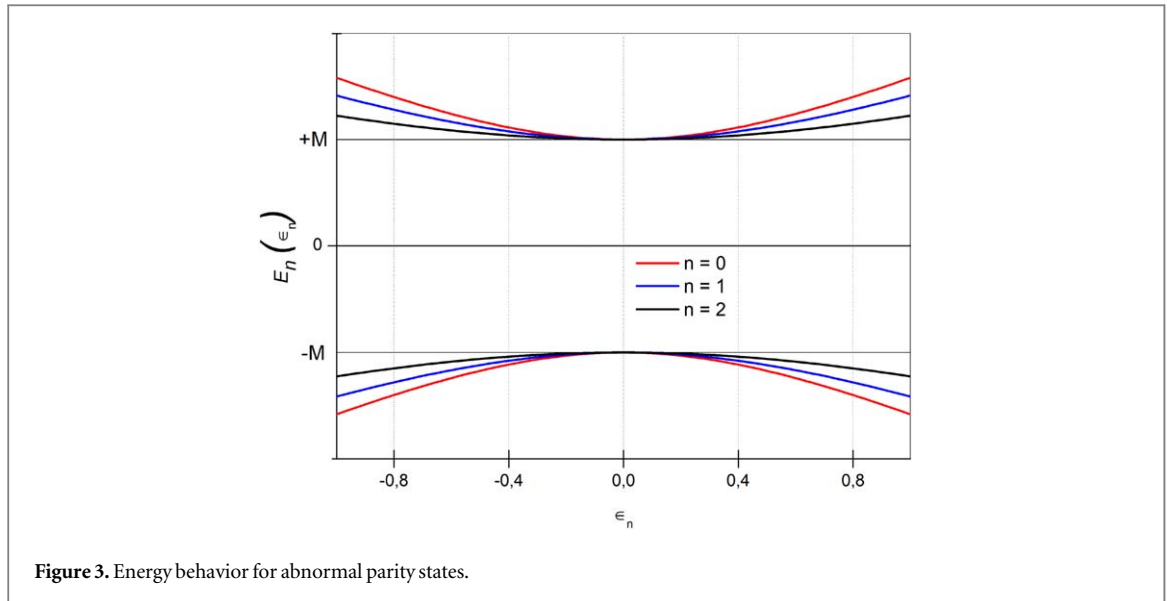


Figure 3. Energy behavior for abnormal parity states.

These eigenvalues can be written as

$$E_n = \pm \sqrt{M^2 + \epsilon_n^2}, \quad \text{where} \quad \epsilon_n^2 = z_r^2 - \left( \frac{x_r z_r}{x_r + n + 2} \right)^2, \quad n = 0, 1, 2, \dots \quad (27)$$

In this case, energy levels do not penetrate the  $E < |M|$  region, so there are no bound states and Klein’s paradox. In figure 3 you can see a graph of (27) for the three lowest values of  $n$ .

Furthermore, the corresponding expression for the  $H_{0,n}(r)$  components is obtained from (17):

$$H_{0,n}(r) = r^{x_r+2} \exp \left[ -r \sqrt{M^2 - E_n^2 + z_r^2} - \frac{y_r}{r} \right] \sum_{m=0}^n c_m r^m. \quad (28)$$

Recall that the potential parameters in both relations (27) and (28) must satisfy the constraint given in (19).

### 3.3.2. $\phi(r)$ solutions

In a completely analogous way we determine the values of the parameters in the case of the  $\phi(r)$  component, comparing (7) and (8),

$$\begin{aligned} \alpha &= E_n^2 - M^2 - z_r^2, & \beta &= -(2x_r + 1)z_r, \\ \omega &= -(j(j + 1) + x_r^2 + 2y_r z_r) \\ \delta &= (1 - 2x_r)y_r, & \sigma &= -y_r^2, \end{aligned} \quad (29)$$

which, together with (13), give the energy relation as

$$(2x_r + 1)z_r = -(2n + 2x_r + 1)\sqrt{M^2 - E_n^2 + z_r^2}, \quad (30)$$

which does not have a real energy spectra, being the spinor component

$$\phi_n(r) = r^{x_r+\frac{1}{2}} \exp \left[ -r \sqrt{M^2 - E_n^2 + z_r^2} - \frac{y_r}{r} \right] \sum_{m=0}^n c_m r^m. \quad (31)$$

Note that, as in the previous cases, the potential parameters in equations (30) and (31) must satisfy the constraint given in (19).

## 4. Heun forms and a class of interactions

The Heun differential equation [44] can be considered a generalization of other special functions in mathematical physics. This equation can be written in a form that resembles Schrödinger’s, namely  $\psi''(r) + V(r)\psi(r) = 0$ , with

$$V(r) = \frac{A_1}{r} + \frac{A_2}{r - 1} + \frac{A_3}{r - a} + \frac{A_4}{r^2} + \frac{A_5}{(r - 1)^2} + \frac{A_6}{(r - a)^2}, \quad (32)$$

in which all parameters except  $r$  are constants chosen simply to show the form of the equation. As already said, special cases of this equation produce other special functions of mathematical physics, including hypergeometric

ones, this being a topic of current interest [44, 51, 53]. In the next subsection, we give the general idea by which some important interactions of physics can be investigated analytically in terms of Heun functions or other simpler special functions. However, it is worth mentioning that each interaction can be the subject of an extensive study separately, since both the mathematical and physical structure of the problem vary from one field to another.

#### 4.1. Generalized cornell interaction and bi-confluent Heun equation

The Cornell potential includes linear and Coulomb terms This is an attractive case in particle physics since it has a confining linear term in addition to the Coulomb interaction, and has been used to study bound states [54]. In its basic form, the resulting equation in the DKP formalism can be investigated simply as it appears in the form of a non-relativistic three-dimensional harmonic term, the solutions of which have been known for decades [55–57]. The generalization of the interaction with a constant extra term appears in the form of bi-confluent Heun function with the effective Hamiltonian [44, 51]

$$V(r) = A_1 r + A_2 r^2 + \frac{A_{-1}}{r} + \frac{A_{-2}}{r^2} + k. \quad (33)$$

It is interesting to note that the linear term is normally called the DKP oscillator, as it results in a harmonic-like form due to the square term, resembling the non-relativistic harmonic oscillator.

#### 4.2. Coulomb plus soft-core Coulomb interaction and confluent-Heun equation

Mehta, a long time ago, proposed a change in the denominator of the Coulomb interaction [58], which is called the truncated or soft-core Coulomb problem. Although the soft-core potential was originally used in particle and nuclear physics, its applications now go much further and the potential is of great interest and application in laser physics, optics, etc [59, 60]. The combination of ordinary and soft-core Coulomb terms, that is, the interaction  $\frac{a_{-1}}{r} + \frac{A_{-1}}{r-\beta}$ , takes the form of confluent-Heun differential equation [44, 51, 53], with

$$V(r) = \frac{A}{r} + \frac{B}{r-1} + \frac{C}{r^2} + \frac{D}{(r-1)^2} + k. \quad (34)$$

The soft-core Coulomb interaction can also be investigated on its own within this approach.

#### 4.3. Kratzer-like interaction and double-confluent Heun equation

In the previous section of this work we have investigated the Kratzer-type potential with the Lie algebraic approach. However, the problem can also be investigated in terms of the double-confluent Heun functions [44, 51, 53]. By making a simple change of variable, the radial equations with our Kratzer-like interaction, appear in the equivalent form of the doubly-confluent Heun equation, with

$$V(r) = \frac{A_{-1}}{r} + \frac{A_{-2}}{r^2} + \frac{A_{-3}}{r^3} + \frac{A_{-4}}{r^4} + k. \quad (35)$$

whose solutions can be written in the form [44, 51, 53]

#### 4.4. Yukawa and exponential-type interactions

In dealing with exponential-type interactions, it is not possible to present analytical solutions unless we make some approximations. Obviously, this is due to the simultaneous presence of inverse and exponential terms. One way to report general solutions is the so-called Pekeris-type approximation, which considers some exponential approximation for the inverse terms, or vice versa [61, 62]. However, this is very rough since the approximation works only for a very limited range and cannot be very competitive with powerful numerical techniques. Nevertheless, there are papers that report analytical solutions with such approaches for various equations of quantum mechanics.

## 5. Conclusion

The spin one DKP equation, unlike other wave equations in quantum mechanics, has not been discussed extensively in the literature. There may be several reasons for this, including the rather complicated structure of the equation, which usually appears in the form of ten components. The analysis of the DKP equation in this case must be done very carefully due to the existence of the so-called unnatural or abnormal parity states. In general, the energy of the natural parity and unnatural parity cases show the relationship between the states with  $S = 0$  and  $|S| = 1$ .

In this work, we consider a Kratzer-type potential to show that abnormal parity states cannot represent bound states for this choice. This highlights the need for careful study of various parity states in related studies.

This could be a good starting point for future research on relativistic vector bosons with more complex physical problems, since a non-minimal vector interaction has been considered, and the problem considered goes beyond what is frequently addressed in the literature. More precisely, articles in this field have so far focused mainly on the so-called DKP oscillator, which is, in fact, a linear interaction whose structure resembles the non-relativistic Schrödinger equation with the harmonic term and possesses exact solutions. Here, we have gone beyond an exactly solvable problem and considered a conditionally exact problem whose exact solution can be reported only if the appropriate conditions among the parameters are met, as shown in figure 2.

To solve the DKP equation with this interaction we have used the powerful Lie algebraic approach and reported the general solution.

We also discuss other classes of interactions, including the Cornell, soft-core, exponential and trigonometric, which are not only pedagogically important to graduates, but also physically motivating and applicable in various fields. We also comment on the energy of various parity states that may, in some way, indicate the absence of Klein's historical paradox.

We hope that with the fairly detailed conceptualization and calculations performed here, the DKP equation will be considered as a basis for studying spin one ions, molecules, hadrons, etc. As a final point, let us mention that it seems that the analysis of the non-Hermitian structure of the equation is very deficient in the field and studies in this direction could open quite new and deep horizons.

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## Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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