HALF-RANGE ORTHOGONALITY THEOREM FOR THERMAL NEUTRON TRANSPORT

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ABSTRACT

It is shown that the eigenfunctions of the energy dependent transport equation with isotropic scattering degenerate kernel have a half-range orthogonality property. Adjoint functions are expressed in terms of the H matrix introduced by Pahor.

1 Introduction

One of the difficulties in the application of the singular eigenfunction expansion method to energy-dependent problems has been that a suitable half-range orthogonality relation concerning the eigenfunctions is not as readily available as in the full-range case. Though the principle of invariance has been successfully applied to solve half-range problems in several models[1-5] the approach is not as convenient as may be desired. Siewert[6] has shown that the eigenfunctions of the two group equation of radiative transfer possess a half-range orthogonality property. Similar half-range orthogonality theorems have been proved in two group transport theory by Siewert and Ishiguro[7] for the case of isotropic scattering and by Ishiguro[8] for the case of linearly anisotropic scattering. The adjoint functions in those cases are expressed in terms of $\underset{\sim}{H}$ matrices which may be obtained from the principle of invariance; however, implications of the established theorems are more general than the principle of invariance.

In the present paper we wish to establish a half-range orthogonality theorem concerning the energy-dependent transport equation for thermal neutrons with isotropic scattering degenerate kernel.

Thus we consider the equation[3]

$$\left[\mu \ell(v) \frac{\partial}{\partial x} + 1\right] \Psi(x, v, \mu) = \frac{1}{2} \int_0^\infty \int_{-1}^{1} f_o(v' \to v)\, \Psi(x, v', \mu')\, \ell(v')\, d\mu' dv' \qquad (1)$$

where the function $\Psi(x, v, \mu)$ is the neutron collision density per unit speed interval and per unit μ interval, the mean free path $\ell(v)$ and the distance x are measured in units of the maximum value of the mean free path. The neutron speed v is measured in units of the most probable speed in the Maxwellian distribution. It is assumed that the mean free path is a monotonic function increasing from $\ell(o) = o$ to $\ell(\infty) = 1$.

By introducing a new variable

$$\xi = \mu\, \ell(v) \qquad (2)$$

Eq. (1) becomes

$$\left[\xi \frac{\partial}{\partial x} + 1\right] \psi(x, v, \xi) = \frac{1}{2} \int_{-1}^{1} d\xi' \int_{v_{\xi'}}^{\infty} f_o(v' \to v) \psi(x, v', \xi') \, dv' \tag{3}$$

with

$$\Psi(x, v, \mu) = \psi(x, v, \xi) \tag{4}$$

and the function v_ξ being defined by the relation

$$|\xi| = \ell(v_\xi). \tag{5}$$

We consider here a kernel written as

$$f_o(v' \to v) = v^3 M(v) \tilde{\underset{\sim}{u}}(v) \underset{\sim}{u}(v') \tag{6}$$

$$M(v) = \pi^{-3/2} \exp(-v^2) \tag{7}$$

where $\underset{\sim}{u}(v)$ is a column vector and a superscript tilde is used to denote the transpose operation.

The general solution to Eq. (3) has been obtained[3,9]. Here we would like only to summarize the results in order to establish our notations.

Seeking the solution of the form

$$\psi(x, v, \xi) = \phi(v, \xi, \nu) \exp(-x/\nu) \tag{8}$$

we find a continuum vector eigenfunction

$$\underset{\sim}{\phi}(v, \xi, \nu) = v^3 M(v) \tilde{\underset{\sim}{u}}(v) \underset{\sim}{F}(\xi, \nu), \nu \epsilon (-1, 1) \tag{9}$$

and discrete eigenfunctions

$$\phi(v, \xi, \pm\nu_j) = v^3 M(v) \tilde{\underset{\sim}{u}}(v) \underset{\sim}{F}(\xi, \pm\nu_j) \underset{\sim}{a}(\nu_j), j = 1,2, \ldots M. \tag{10}$$

Here

$$\underset{\sim}{F}(\xi, \nu) = \frac{\nu}{2} \frac{P}{\nu - \xi} \underset{\sim}{E} + \underset{\sim}{G}^{-1}(\nu) \underset{\sim}{\lambda}(\nu) \delta(\nu - \xi) \tag{11}$$

with P referring to the principal value of the integrals and $\underset{\sim}{E}$ being the unit matrix. The symmetric matrix $\underset{\sim}{G}(\nu)$ is given by

$$\underset{\sim}{G}(\nu) = \int_{v_\nu}^{\infty} v^3 M(v) \underset{\sim}{u}(v) \tilde{\underset{\sim}{u}}(v) \, dv \tag{12}$$

and the discrete eigenvalues $\pm\nu_j$ are the roots of the equation

$$\Lambda(z) = \det \underset{\sim}{\Lambda}(z) = 0 \tag{13}$$

where

$$\underset{\sim}{\Lambda}(z) = \underset{\sim}{E} - \frac{z}{2} \int_{-1}^{1} \underset{\sim}{G}(\xi) \frac{d\xi}{z - \xi} \tag{14}$$

The number of discrete eigenvalue pairs is denoted by M and it is assumed, in the present paper, that $\Lambda(\infty) \neq 0$

The boundary values of the dispersion matrix are

$$\underset{\sim}{\Lambda}^{\pm}(\nu) = \underset{\sim}{\lambda}(\nu) \pm \frac{\pi i \nu}{2} \underset{\sim}{G}(\nu), \nu \epsilon (-1, 1) \tag{15}$$

and the vector $a(\nu_j)$ satisfies the equation

$$\underset{\sim}{\Lambda}(\nu_j)\, \underset{\sim}{a}(\nu_j) = \underset{\sim}{O}, j = 1,2, \ldots, M \tag{16}$$

2 The $\underset{\sim}{H}$ Matrix

Pahor[3], considering the half space albedo problem and applying the principle of invariance, has introduced the $\underset{\sim}{H}$-matrix which satisfies the nonlinear integral equation

$$\underset{\sim}{\tilde{H}}(z) \left[\underset{\sim}{E} - \frac{z}{2} \int_{0}^{1} \underset{\sim}{H}(\xi)\, \underset{\sim}{G}(\xi) \frac{d\xi}{\xi + z} \right] = \underset{\sim}{E}, z \notin (-1, 0). \tag{17}$$

The $\underset{\sim}{H}$ matrix also satisfies the equations

$$\underset{\sim}{H}(z)\underset{\sim}{\Lambda}(z) = \underset{\sim}{E} - \frac{z}{2} \int_{0}^{1} \underset{\sim}{H}(\xi)\, \underset{\sim}{G}(\xi) \frac{d\xi}{z - \xi}, z \notin [-1, 1] \tag{18a}$$

$$\underset{\sim}{H}(\nu)\underset{\sim}{\lambda}(\nu) = \underset{\sim}{E} - \frac{\nu}{2} P \int_{0}^{1} \underset{\sim}{H}(\xi)\, \underset{\sim}{G}(\xi) \frac{d\xi}{\nu - \xi}, \nu \epsilon (0, 1). \tag{18b}$$

Constraining the $\underset{\sim}{H}$ matrix such that

$$\det \left| \underset{\sim}{E} - \int_{0}^{1} \underset{\sim}{H}(\xi)\, \underset{\sim}{G}(\xi)\, \underset{\sim}{F}(\xi, \nu_j)\, d\xi \right| = 0, j = 1,2,\ldots, M \tag{19}$$

ensures that Eqs. (17) and (18) have a unique solution which is analytic in the complex plane cut from -1 to 0 except at $z = -\nu_j$ where $\underset{\sim}{H}(z)$ has a simple pole.

3. Half-Range Orthogonality

We would now like to establish the half range orthogonality theorem: (hereafter $\nu, \nu' > 0$).

Theorem: the eigenfunctions $\underset{\sim}{\phi}(v,\xi,\nu)$, $\nu\epsilon(0,1)$, and $\phi(v,\xi,\nu_j)$, $j = 1,2, \ldots, M$, are orthogonal on the half-range $\xi\epsilon(0,1)$ in the sense that

$$\int_{0}^{\infty} \int_{0}^{1} a(v, \xi, \nu_j)\phi(v, \xi, \nu_k)\xi d\xi dv = 0, j \neq k \tag{20a}$$

$$\int_0^1 \int_{v\xi}^{\infty} \theta(v,\xi,\nu_j)\, \underset{\sim}{\phi}(v,\xi,\nu)\, \xi d\xi dv = \underset{\sim}{O} \tag{20b}$$

$$\int_0^1 \int_{v\xi}^{\infty} \underset{\sim}{\theta}(v,\xi,\nu)\, \phi(v,\xi,\nu_j)\, \xi d\xi dv = \underset{\sim}{O} \tag{20c}$$

$$\int_0^1 \int_{v\xi}^{\infty} \underset{\sim}{\theta}(v,\xi,\nu)\, \underset{\sim}{\phi}(v,\xi,\nu')\, \xi d\xi dv = \underset{\sim}{O}\,, \nu \neq \nu' \tag{20d}$$

where the adjoint functions are given by

$$\theta(v,\xi,\nu_j) = \underset{\sim}{\tilde{a}}(\nu_j)\, \underset{\sim}{H}^{-1}(\nu_j)\, \underset{\sim}{H}(\xi)\, \underset{\sim}{\tilde{F}}(\xi,\nu_j)\, \underset{\sim}{u}(v) \tag{21a}$$

$$\underset{\sim}{\theta}(v,\xi,\nu) = \underset{\sim}{H}^{-1}(\nu)\, \underset{\sim}{H}(\xi)\, \underset{\sim}{\tilde{F}}(\xi,\nu)\, \underset{\sim}{u}(v). \tag{21b}$$

Proof: If we substitute Eqs. (21b), (9) and (11) into Eq. (20d) we obtain, after some partial fraction analysis and use of Eq. (12),

$$\int_0^1 \int_{v\xi}^{\infty} \underset{\sim}{\theta}(v,\xi,\nu)\, \underset{\sim}{\phi}(v,\xi,\nu')\, \xi d\xi dv$$

$$= \frac{\nu\nu'}{2(\nu'-\nu)}\, \underset{\sim}{H}^{-1}(\nu) \left[\frac{\nu}{2}\int_0^1 \underset{\sim}{H}(\xi)\underset{\sim}{G}(\xi)\, \frac{P}{\nu-\xi}\, d\xi\right.$$

$$\left. - \frac{\nu'}{2}\int_0^1 \underset{\sim}{H}(\xi)\, \underset{\sim}{G}(\xi)\, \frac{P}{\nu'-\xi}\, d\xi + \underset{\sim}{H}(\nu)\underset{\sim}{\lambda}(\nu) \quad \underset{\sim}{H}(\nu')\underset{\sim}{\lambda}(\nu')\right]. \tag{22}$$

The use of Eq.(18b) proves Eq. (20d), and the other cases can be proved in a similar manner using Eq. (16) as well as Eqs. (18).

Finally we would like to list the normalization integrals[3,9]

$$\int_0^1 \int_{v\xi}^{\infty} \theta(v,\xi,\nu_j)\phi(v,\xi,\nu_j)\xi d\xi dv = \frac{1}{2}\, \nu_j^2\, \underset{\sim}{\tilde{a}}(\nu_j)\underset{\sim}{\Lambda}'(\nu_j)\underset{\sim}{a}(\nu_j) \tag{23a}$$

$$\int_0^1 \int_{v\xi}^{\infty} \underset{\sim}{\theta}(v,\xi,\nu)\underset{\sim}{\phi}(v,\xi,\nu')\xi d\xi dv = \nu\underset{\sim}{\Lambda}^{+}(\nu)\, \underset{\sim}{G}^{-1}(\nu)\, \underset{\sim}{\Lambda}^{-}(\nu)\xi(\nu-\nu') \tag{23b}$$

and the cross product integrals

$$\int_0^1 \int_{v\xi}^{\infty} \theta(v,\xi,\nu_j)\, \phi(v,\xi,-\nu_k)\xi d\xi dv = \frac{\nu_j\,\nu_k}{2(\nu_j+\nu_k)}\, \underset{\sim}{\tilde{a}}(\nu_j)\underset{\sim}{H}^{-1}(\nu_j)\underset{\sim}{\tilde{H}}^{-1}(\nu_k)\underset{\sim}{\tilde{a}}(\nu_k) \tag{24a}$$

$$\int_0^1 \int_{v\xi}^{\infty} \theta(v,\xi,\nu_j)\, \underset{\sim}{\phi}(v,\xi,-\nu)\,\xi\, d\xi\, dv = \frac{\nu\nu_j}{2(\nu+\nu_j)}\; \underset{\sim}{\tilde{a}}(\nu_j)\, \underset{\sim}{H}^{-1}(\nu_j)\, \underset{\sim}{\tilde{H}}^{-1}(\nu) \tag{24b}$$

$$\int_0^1 \int_{v\xi}^{\infty} \underset{\sim}{\theta}(v,\xi,\nu)\, \phi(v,\xi,-\nu_j)\,\xi\, d\xi\, dv = \frac{\nu\nu_j}{2(\nu+\nu_j)}\; \underset{\sim}{H}^{-1}(\nu)\, \underset{\sim}{\tilde{H}}^{-1}(\nu_j)\, \underset{\sim}{a}(\nu_j) \tag{24c}$$

$$\int_0^1 \int_{v\xi}^{\infty} \underset{\sim}{\theta}(v,\xi,\nu)\, \underset{\sim}{\phi}(v,\xi,-\nu')\,\xi\, d\xi\, dv = \frac{\nu\nu'}{2(\nu+\nu')}\; \underset{\sim}{H}^{-1}(\nu)\, \underset{\sim}{\tilde{H}}^{-1}(\nu'). \tag{24d}$$

RESUMO

É demonstrado que as autofunções da equação de transporte dependente da energia com um Kernel de espalhamento isotrópico degenerado apresenta a propriedade de ortogonalidade no semi-intervalo.

As funções adjuntas são expressas em têrmos da matriz $\underset{\sim}{H}$ introduzida por Pahor.

RESUME

On a prouvé que les fonctions propes de l'equation de transport dependant de l'énergie avec Kernel de ralentissement isotropique dégeneré ont une proprieté d'orthogonalité à demiporté. Les fonctions adjointes sont établies en règle de la matrice $\underset{\sim}{H}$ introduite par Pahor.

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