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THE FINITE ELEMENT RESPONSE MATRIX METHOD

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ABSTRACT

A new technique is developed with an alternative formulation of the response matrix method implemented with the finite element scheme. Two types of response matrices are generated from the Galerkin solution to the weak form of the diffusion equation subject to an arbitrary current and source. The piecewise polynomials are defined in two levels, the first for the local (assembly) calculations and the second for the global (core) response matrix calculations. This finite element response matrix technique was tested in two 2-dimensional test problems, 2D-IAEA benchmark problem and Bibis benchmark problem, with satisfatory results. The computational time, whereas the current code is not extensively optimized, is of the same order of the well estabilished coarse mesh codes. Furthermore, the application of the finite element technique in an alternative formulation of response matrix method permits the method to easily incorporate additional capabilities such as treatment of spatially dependent cross-sections, arbitrary geometrical configurations, and highly heterogeneous assemblies.

INTRODUCTION

The finite element method (FEM) has been widely used in structural analysis[15] and recently its application to the reactor analysis has increased considerably[14]. Some of attractive features of FEM is its flexibility in the choice of appropriate basis functions in arbitrary degrees of approximations allowing the method to be applied to quite irregular geometries including local refinements of local heterogeneities. In addition the theoretical foundations of the FEM are well estabilished[7] and definite analytic error bounds can be predicted for most applications of interest.

In the present work an alternative formulation of the response matrix method[12] is implemented using the finite element method in order to perform relatively fast coarse-mesh calculations with satisfatory accuracy. The basic idea of the present approach is to construct for each coarse-mesh, two types of response matrices applying the finite element method to solve the fixed source diffusion problem subject to an arbitrary current in the boundary of the coarse-mesh. First response matrix gives the outgoing partial current due to the diffusion of the incoming current and the second the outgoing current diffusing from the source within the coarse meshes (inscatter, fission or external source). These local response matrix are projected on the independent global basis functions and the global solution is obtained by balancing the partial currents defined on this global basis functions. The transformations between the local and global current distributions (and vice-versa) are performed in the global iteration procedure, yielding a global partial current distribution which takes into account local heterogeneities which are capable of being resolved by the finer grid used for the local response matrix calculations but which would not be resolved by the grid at the global level. The utilization of FEM to implement the response matrix method allows the method to be capable of treating the spatially dependent cross sections, feature which is very convenient for burn-up calculations[6]. Local quantities such as flux and power distributions are explicitly defined over the entire core even in heterogeneous assemblies (e.g. water hole or poison pins) by the FEM in the local level calculations. Furthermore the use of two types of response matrices, for currents and sources, make the multigroup eigenvalue calculations straightforward and conventional solution techniques can be applied.

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Response Matrix Method

The alternative formulation of the response matrix method suitable for finite element implementation is described following closely the standard formulation of response matrix method from Weiss and Lindahl[13].

Assume the domain V in which the solution for the neutron diffusion equation is sought be divided in N subdomains V_i, i = 1, . . . , N, and they will be referred to as coarse-meshes. Each coarse-mesh V_i is bounded by a piecewise smooth boundary S_i with the outward directed normal vector $\underline{n}(\underline{r}_s)$, where $\underline{r}_s$ is the position vector on S_i.

It is assumed that an arbitrary accurate solution for the inhomogeneous neutron diffusion equation ($D_i \nabla^2 \phi_i - \Sigma_{ai}\phi_i + Q_i = 0$) can be found in coarse-mesh V_i subject to the boundary condition on S_i, corresponding to the irradiation of S_i by an arbitrary current $J_i^-(\underline{r}_s)$. The emerging partial current $J_i^+(\underline{r}_s)$ on S_i can be obtained from the solution for the neutron flux.

Because of the linearity of the neutron diffusion equation the relationship between $J_i^+(\underline{r}_s)$ and $J_i^-(\underline{r}_s)$ and $Q_i(\underline{r}_s)$ can be concisely expressed by a linear transformation,

$$J_i^+(\underline{r}_s) = \oint_{S_i} R_i^S(\underline{r}'_s \to \underline{r}_s)\, J_i^-(\underline{r}'_s)\, d\underline{r}'_s + \int_{V_i} R_i^V(\underline{r} \to \underline{r}_s)\, Q_i(\underline{r})\, d\underline{r} \qquad (1)$$

where,

$R_i^S(\underline{r}'_s \to \underline{r}_s)$ is the response kernel for outgoing partial current due to diffusion of incoming current,

$R_i^V(\underline{r} \to \underline{r}_s)$ is the response kernel for outgoing current due to diffusion of neutron produced within V_i (e.g. inscatter, fission, source),

$Q_i(\underline{r})$ is the total production inside V_i due to fission, inscatter, source.

Defining $H_{ij}(\underline{r}'_s \to \underline{r}_s)$, i, j = 1, . . . N, as the probability that a neutron leaving the surface S_j at $\underline{r}'_s$ reaches $\underline{r}_s$ on the surface S_i, the inward partial current $J_i^-(\underline{r}'_s)$ for the coarse mesh V_i can be related to the outward partial current $J_j^+(\underline{r}_s)$ from the coarse meshes V_j, j = 1, . . . , N, by the compatibility equation,

$$J_i^-(\underline{r}_s) = \sum_{j=1}^{N} H_{ij}(\underline{r}'_s \to \underline{r}_s)\, J_j^+(\underline{r}'_s),\ i = 1, \ldots, N. \qquad (2)$$

In most cases of interest this compatibility equation expresses continuity of partial current normally imposed by simple re-indexing or permutation. For some special problems additional calculations are required like in spent fuel storage problems.

As a byproduct of the response kernels calculations, the neutron flux inside the coarse mesh V_i can be given by

$$\Phi_i(\underline{r}) = \oint_{S_i} M_i^S(\underline{r}_s \to \underline{r})\, J_i^-(\underline{r}_s)\, d\underline{r}_s + \int_{V_i} M_i^V(\underline{r}' \to \underline{r})\, Q_i(\underline{r}')\, d\underline{r}' \qquad (3)$$

where $M_i^s(\underline{r}_s \to \underline{r})$ and $M_i^v(\underline{r}' \to \underline{r})$ are response kernels yielding the flux due to an incoming current and neutron source, respectively.

Thus the difference between the alternative formulation of response matrix Equations (1) – (3) and the standard response matrix Equations[13] is the definition of source response kernels, $R_i^v(\underline{r}' \to \underline{r})$ and $M_i^v(\underline{r}' \to \underline{r})$. These kernels will be shown to be very convenient for eigenvalue calculation by conventional solution techniques.

Numerical Approximation

The response matrix Equations (1)–(3) can be cast in a numerically manageable form by expanding the partial currents, fluxes and sources in appropriate polynomials.

Let the coarse meshes V_i, $i = 1, \ldots, N$, contain L interior nodes and K boundary nodes on S_i, and define the polynomials $\Psi_l(\underline{r})$, $l = 1, \ldots, L$, with unitary value at the node l and zero at the remaining nodes, and polynomials $\Psi_{sk}(\underline{r}_s)$, $k = 1, \ldots, K$, with unitary value at the node k and zero at the remaining nodes.

Then the partial currents, the neutron flux and the neutron source can be approximated for each coarse mesh V_i by an expansion of the form

$$J_i^{\pm}(\underline{r}_s) = \sum_{k=1}^{K} J_{i_k}^{\pm} \Psi_{s_k}(\underline{r}_s) \qquad , \underline{r}_s \in S_i ,$$

$$\Phi_i(\underline{r}) = \sum_{l=1}^{L} \Phi_{i_l} \Psi_l(\underline{r}) \qquad , \underline{r} \in V_i ,$$

$$\text{and } Q_i(\underline{r}) = \sum_{l=1}^{L} Q_{i_l} \Psi_l(\underline{r}) \qquad , \text{for } i = 1, \ldots, N$$

where $J_{i_k}^{\pm}$, Φ_{i_l}, Q_{i_l} are the partial current value at the node k on the boundary S_i, the neutron flux value and source value at the node l in the coarse mesh V_i, respectively.

Inserting these expansions into the response matrix Equations (1)–(3) and applying the Galerkin weighted residual technique the matrix equations for the expansion coefficients are obtained as

$$\underline{J}_i^+ = \underline{\underline{R}}_i^s \cdot \underline{J}_i^- + \underline{\underline{R}}_i^- \cdot \underline{Q}_i$$

$$\underline{J}_i^- = \sum_{j=1}^{N} \underline{\underline{H}}_{ij} \cdot \underline{J}_j^+$$

$$\underline{\Phi}_i = \underline{\underline{M}}_i^s \cdot \underline{J}_i^- + \underline{\underline{M}}_i^v \cdot \underline{Q}_i , \qquad i = 1, \ldots, N,$$

where,

$$\underline{J}_i^{\pm} = \text{col}\,(J_{i_1}^{\pm}, J_{i_2}^{\pm}, \ldots, J_{i_K}^{\pm})$$

$$\underline{\Phi}_i = \text{col}\,(\Phi_{i_1}, \Phi_{i_2}, \ldots, \Phi_{i_L})$$

$$\underline{Q}_i = \text{col}\,(Q_{i_1}, Q_{i_2}, \ldots, Q_{i_L})$$

and the matrices $\underline{\underline{R}}_i^v$, $\underline{\underline{R}}_i^s$, $\underline{\underline{H}}_{ij}$, $\underline{\underline{M}}_i^v$, $\underline{\underline{M}}_i^s$ are generated basically from inner products of appropriate basis functions and the kernels $R_i^v(\underline{r} \to \underline{r}_s)$, $R_i^s(\underline{r}'_s \to \underline{r}_s)$, $H_{ij}(\underline{r}'_s \to \underline{r}_s)$, $M_i^v(\underline{r}' \to \underline{r})$ and $M_i^s(\underline{r}_s \to \underline{r})$, pectively.

A compact representation of the response matrix equations for the domain V in which the solution or the diffusion equations is sought can be obtained with the following definitions,

$$\underline{J}^+ = \text{col}\,(\underline{J}_1^+, \underline{J}_2^+, \ldots, \underline{J}_N^+)$$

$$\underline{\Phi} = \text{col}\,(\underline{\Phi}_1, \underline{\Phi}_2, \ldots, \underline{\Phi}_N)$$

$$\underline{Q} = \text{col}\,(\underline{Q}_1, \underline{Q}_2, \ldots, \underline{Q}_N)$$

and $\underline{\underline{R}}^s$, $\underline{\underline{R}}^v$, $\underline{\underline{M}}^s$, $\underline{\underline{M}}^v$ as NxN diagonal block matrix with elements $\underline{\underline{R}}_i^s$, $\underline{\underline{R}}_i^v$, $\underline{\underline{M}}_i^s$, $\underline{\underline{M}}_i^v$, $i = 1, \ldots, N$, respectively, and $\underline{\underline{H}}$ as NxN block matrix with elements $\underline{\underline{H}}_{ij}$, $i, j = 1, \ldots, N$.

Then the response matrix equations can be succintly expressed as

$$\underline{J}^+ = \underline{\underline{R}}^s \cdot \underline{J}^- + \underline{\underline{R}}^v \cdot \underline{Q} \tag{4}$$

$$\underline{J}^- = \underline{\underline{H}} \cdot \underline{J}^+ \tag{5}$$

$$\underline{\Phi} = \underline{\underline{M}}^s \cdot \underline{J}^- + \underline{\underline{M}}^v \cdot \underline{Q} \tag{6}$$

Solution Algorithm

The solution of the response matrix Equations (4)–(6) for a fixed source problem is immediately obtained from Equations (4) and (5) as:

$$\underline{J}^+ = \underline{\underline{R}}^s \cdot \underline{\underline{H}} \cdot \underline{J}^+ + \underline{\underline{R}}^v \cdot \underline{Q}$$

but for practical problems the direct inversion of $(\underline{\underline{I}} - \underline{\underline{R}}^s \cdot \underline{\underline{H}})$ is prohibitively expensive and the iterative method[7] must be applied as:

$$\underline{J}^{+(t)} = \underline{\underline{R}}^s \cdot \underline{\underline{H}} \cdot \underline{J}^{+(t-1)} + \underline{\underline{R}}^v \cdot \underline{Q}, \quad t = 1, 2, \ldots \tag{7}$$

The response matrix method formulated in the previous section is in the suitable form to treat the eigenvalue problems, where the neutron source, in one group, is given by:

$$\underline{Q} = \frac{1}{k_{eff}} \underline{\underline{F}} \cdot \underline{\Phi}$$

where $\underline{\underline{F}}$ matrix is composed of fission neutron production cross section and keff is the multiplication factor, the largest eigenvalue to be determined. Therefore the eigenvalue problem can be solved by the standard outer (source) iteration and it is given by the algorithm

$$\underline{J}^{+(t)} = \underline{\underline{R}}^{s} \cdot \underline{\underline{H}} \cdot \underline{J}^{+(t-1)} + \underline{\underline{R}}^{v} \cdot \underline{Q}^{(r-1)}, \quad t = 1, 2 \ldots \tag{8.a}$$

$$\underline{Q}^{(r)} = \frac{1}{keff^{(r-1)}} \underline{\underline{F}} \cdot (\underline{\underline{M}}^{s} \cdot \underline{\underline{H}} \cdot \underline{J}^{+(t)} + \underline{\underline{M}}^{v} \cdot \underline{Q}^{(r-1)}), \; r = 1, 2, \ldots \tag{8.b}$$

where t denotes the inner (fixed source) iterations count and r denotes the outer (source) iterations count.

Extension for multigroup eigenvalue problem is immediate.

Calculation of the Response Matrices with FEM

The response matrix equations can be solved whenever the response matrices are computable. In particular the solution by the FEM for the neutron diffusion equation in the coarse mesh as described below presents some attractive aspects, e.g. the freedom in the geometry choice, the capability to include heterogeneous assemblies or spatially dependent cross-sections. In addition the choice of suitable basis functions is facilitated since the definitions are local and in piecewise manner and the natural boundary conditions are not necessary to be satisfied by the basis functions in the weak formulation.

The diffusion problem to be solved inside the coarse mesh V_n is

$$\nabla D_n(\underline{r}) \nabla \phi_n(\underline{r}) + \Sigma_{an}(\underline{r}) \phi_n(\underline{r}) = q_n(\underline{r}), \quad \underline{r} \in V_n, \; n = 1, \ldots, N \tag{9.a}$$

where

$D_n(\underline{r})$ is the diffusion coefficient,

$\Sigma_{an}(\underline{r})$ is the total absorption cross section,

$q_n(\underline{r})$ is the arbitrary neutron sourse,

subject to the irradiation of an arbitrary inward partial current $J_n^-(\underline{r_s})$ on the boundary S_n,

$$j_n^-(\underline{r_s}) = \frac{1}{4}\phi_n(\underline{r_s}) + \frac{1}{2} D_n(\underline{r_s}) \nabla \phi_n(\underline{r}) \Big|_{\underline{r} = \underline{r_s}} \cdot \underline{n}(\underline{r_s}), \; \underline{r_s} \in S_n, \; n = 1, \ldots, N \tag{9.b}$$

where $\underline{n}(\underline{r_s})$ is the vector normal to the surface S_n, and also requiring the continuity of flux and current to be satisfied inside the coarse mesh V_n.

The solution is assumed to be in the energy space[7] H_E defined as:

$$H_E = \{ f(\underline{r}) \in C^0 \Big| \int_{V_n} [\nabla j(\underline{r}) + j(\underline{r})]^2 d\underline{r} < \infty \},$$

and defining $\psi_n(\underline{r})$, an arbitrary element of space H_E of trial functions, the weak formulation of problem(9) corresponds to finding a solution $\phi_n(\underline{r})$ in H_E for

$$\int_{V_n} D_n(\underline{r})\nabla\phi_n(\underline{r})\nabla\psi_n(\underline{r})d\underline{r} + \int_{V_n}\Sigma_{an}\phi_n(\underline{r})\psi_n(\underline{r})d\underline{r} + \frac{1}{2}\int_{S_n}\oint_{S_n}\phi_n(\underline{r_s})\psi_n(\underline{r_s})d\underline{r_s} =$$

$$= \int_{V_n} q_n(\underline{r})\psi_n(\underline{r})d\underline{r} + \oint_{S_n} j_n^-(\underline{r_s})\psi_n(\underline{r_s})d\underline{r_s}\,;\quad \phi_n(\underline{r}) \in H_E,\ n = 1,\ldots,N, \tag{10}$$

for all $\psi_n(\underline{r}) \in H_E$.

The finite element solution of problem(9) is obtained from the weak formulation by defining the trial functions within a particular finite dimensional subspace S^h contained in H_E.

In order to construct the finite element subspace S^h, let the coarse mesh V_n contain Ne nodes and define $\psi^h_{ni}(\underline{r})$, $i = 1,\ldots,$ Ne, $\psi^h_{ni}(\underline{r}) \in S^h$, piecewise continuous polynomials with unitary value at the node i and zero at the remaining nodes. These polynomials are the basis functions for the subspace S^h, and every member of S^h can be given as a combination of $\psi^h_{ni}(\underline{r})$.

If the solution $\phi^h_n(\underline{r})$ is then sought within the space S^h, $\phi^h_n(\underline{r})$ can be expanded in the basis for S^h as

$$\phi^h_n(\underline{r}) = \sum_{i=1}^{Ne}\phi_{ni}\psi^h_{ni}(\underline{r}) \equiv [\underline{\psi}^h_n(\underline{r})]^T \cdot \underline{\phi}_n \tag{11}$$

where, $\underline{\phi}_n = \mathrm{col}(\phi_1, \phi_2, \ldots, \phi_{Ne})$

$$\underline{\psi}^h_n(\underline{r}) = \mathrm{col}(\psi^h_{n1}(\underline{r}), \psi^h_{n2}(\underline{r}), \ldots, \psi^h_{nNe}(\underline{r})) \tag{12}$$

and requiring the weak form (10) to be valid for this particular aproximation of $\phi_n(\underline{r})$ for all basis functions $\psi^h_{nj}(\underline{r}) \in S^h$, $j = 1,\ldots,$ Ne, the resulting matrix equation is

$$\underline{\underline{A}}_n \cdot \underline{\phi}_n = \underline{q}_n + \underline{j}_n^-, \qquad n = 1,\ldots,N, \tag{12}$$

where, $\underline{\underline{A}}_n = \left[\int_{V_n} D_n(\underline{r})\nabla\psi^h_{ni}(\underline{r})\nabla\psi^h_{nj}(\underline{r})d\underline{r} + \int_{V_n}\Sigma_{an}(\underline{r})\psi^h_{ni}(\underline{r})\psi^h_{nj}(\underline{r})d\underline{r} + \frac{1}{2}\oint_{S_n}\psi^h_{ni}(\underline{r_s})\psi^h_{nj}(\underline{r_s})d\underline{r_s}\right]_{ij}$;

$i, j = 1,\ldots,$ Ne;

$q_{ni} = \int_{V_n} q_n(\underline{r})\psi^h_n(\underline{r})d\underline{r}$; $\qquad \underline{j}_n = 2\oint_{S_n} j_n^-(\underline{r_s})\psi^h_n(\underline{r_s})d\underline{r_s}$

and since $\underline{\underline{A}}_n$ is symmetric positive definite for $D_n(\underline{r}) > 0$ and $\Sigma_{an}(\underline{r}) \geq 0$, the solution for ϕ_n can be determined as

$$\underline{\phi}_n = \underline{\underline{A}}_n^{-1} \cdot \underline{q}_n + \underline{\underline{A}}_n^{-1} \cdot \underline{j}_n^-, \qquad n = 1,\ldots,N. \tag{13}$$

The local response matrices for the coarse mesh V_n is obtained from the diffusion expression for the partial current

$$j_n^+(\underline{r_s}) = \frac{1}{4}\phi_n(\underline{r_s}) - \frac{1}{2}D_n(\underline{r_s})\nabla\phi_n(\underline{r})\Big|_{\underline{r}=\underline{r_s}} \cdot \underline{n}(\underline{r_s});\ n = 1,\ldots,N, \tag{14}$$

by expanding $j_n^+(r_s)$ in the boundary basis function $\psi^h_{sni}(r_s)$, $i = 1, \ldots, Ns$ defined as piecewise polynomials with unitary value at the boundary node i and zero at the remaining boundary nodes,

$$j_n^+(r_s) = \sum_{i=1}^{Ns} j^+_{ni}\, \psi^h_{Sni}(r_s) \equiv [\underline{\psi}^h_{sn}(r_s)]^T \cdot j_n^+, \quad n = 1, \ldots, N, \tag{15}$$

where,

$$\underline{\psi}^h_{s_n}(r_s) = \text{col}\,(\psi_{s_{n_1}}(r_s), \psi_{s_{n_2}}(r_s), \ldots, \psi_{s_{nN_e}}(r_s))$$

$$J^{\pm}_n = \text{col}\,(J^{\pm}_{n_1}, \ldots, J^{\pm}_{nNe}).$$

Inserting the Equations (11), (13) and (15) into the Equation (14), and applying the weighted residual technique, the local response matrix equation is obtained as

$$j_n^+ = R'_{q_n} \cdot q_n + R'_{j_n} \cdot j_n^- \qquad n = 1, \ldots, N, \tag{16}$$

where R'_{q_n} and R'_{j_n} are the resulting local response matrices.

The Equations (13) and (16) together with continuity of partial currents

$$j_n^- = \sum_{n=1}^{N} H'_{nm} \cdot j_m^-, \qquad n = 1, \ldots, N, \tag{17}$$

where H'_{nm} is composed of appropriate inner products of local basis function and of the kernels $H_{nm}(r'_s \rightarrow r_s)$ defined in Equation (2), represents the finite element solution for the diffusion Equation (9) in the coarse meshes. $V_n, n = 1, \ldots, N$.

Global Response Matrices

The solution in the domain V can be obtained by assembling all the individual local response matrices in such a way that in the global solution the number of unknowns is not large compared with other coarse mesh methods. To this end recall the global response matrix equations derived in the Equations (4), (5) and (6), and since $q_n(r)$ and $j_n^-(r_s)$ are quite arbitrary, it is convenient to define these quantities as $Q_n(r)$ and $J_n^-(r_s)$, respectively, and expand in the global basis functions $\Psi_l(r)$, $l = 1, \ldots, L$, and $\Psi_{sk}(r_s)$, $k = 1, \ldots, K$, respectively, as

$$q_n(r) = [\underline{\Psi}(r)]^T \cdot Q_n, \qquad n = 1, \ldots, N,$$

and

$$j_n^-(r_s) = [\underline{\Psi}_s(r_s)]^T \cdot J_n^-, \qquad n = 1, \ldots, N,$$

where,

$$\underline{\Psi}(r) = \text{col}\,(\Psi_1(r), \Psi_2(r), \ldots, \Psi_L(r))$$

and

$$\underline{\Psi}_s(r) = \text{col}\,(\Psi_{s_1}(r_s), \Psi_{s_2}(r_s), \ldots, \Psi_{s_K}(r_s)).$$

Therefore $\underline{q}_n$ and $\underline{j}_n^-$ defined in Equation (12) can be expressed as

$$\underline{q}_n = \underline{\underline{T}}_{qQ} \cdot \underline{Q}_n, \qquad n = 1, \ldots, N, \tag{18.a}$$

and

$$\underline{j}_n^- = \underline{\underline{T}}_{j} \cdot \underline{J}_n^-, \qquad n = 1, \ldots, N, \tag{18.b}$$

where $\underline{\underline{T}}_{qQ}$ and $\underline{\underline{T}}_{jJ}$ are matrices composed of appropriate inner products of the local and global basis functions.

Combining Equations (16) and (18)

$$\underline{j}_n^+ = \underline{\underline{R}}_{q_n} \cdot \underline{Q}_n + \underline{\underline{R}}_{j_n} \cdot \underline{J}_n^-, \qquad n = 1, \ldots, N, \tag{19}$$

which gives the local outgoing current due to an arbitrary global source and an arbitrary global incident current.

The global outgoing current, $J_n^+(r_s)$, cannot be equated to the local current, $j_n^+(r_s)$, because there is no unique way to compute the expansion coefficients $\underline{J}^+$, but we can require the global current to equal the local current in a weighted residual sense, yielding

$$\underline{J}_n^+ = \underline{\underline{T}}_{J_j} \cdot \underline{j}_n^+, \tag{20}$$

and combining Equation (20) with Equation (19) we have the desired global response matrix equations.

$$\underline{J}_n^+ = \underline{\underline{R}}_n^v \cdot \underline{Q}_n + \underline{\underline{R}}_n^- \cdot \underline{J}_n^-; \qquad n = 1, \ldots, N, \tag{21}$$

Where $\underline{\underline{R}}_n^v$ and $\underline{\underline{R}}_n^s$ are the desired composite response matrices consisting of several products and sums of individual matrices.

Similar manipulations yield the global flux equation for coarse mesh V_n.

$$\underline{\psi}_n = \underline{\underline{M}}_n^v \cdot \underline{Q}_n + \underline{\underline{M}}_n^s \cdot \underline{J}_n^-, \qquad n = 1, \ldots, N. \tag{22}$$

Combined with the compatibility equation the global response matrix Equations (21) and (22) can be solved in an iterative manner as described in the Equations (7) and (8).

Numerical Results

In the present work the finite element response matrix method (FERMM) were applied to the two-dimensional problems and the geometry of the coarse mesh was chosen as a rectangle, since most of the practical problems in reactor calculations can be solved in this geometry, but extension to the triangular geometry or other geometries, and extension to three-dimensional problems is immediate and does not impose any conceptual difficulties.

The serendipity elements[7] were chosen as basis functions for both local and global volume basis functions. The global basis functions were defined either in quadratic or in cubic approximations and the basis functions used to generate the response matrices in the local (assembly) calculations for

the coarse meshes were chosen to be quadratic elements with the degree of approximations obtained by varying the number of nodes in a coarse mesh. For special assemblies (e.g. Control rod assemblies) finer grid are possible to be used in the local calculation in order to represent with higher accuracy the actual flux shapes in the assembly. The partial current were expanded in Lagrange polynomials on the boundary in the piecewise manner with discontinuities at the corners in order to allow for the discontinuity of partial current defined by the diffusion theory (Equation 9,b).

The numbering sheme for rectangular coarse meshes resulted in global matrix equations (Equations (4) and (5)) recognizable as a 2-cyclic block-Jacobi matrix equation[8] and the iterative solution applied to its solution is the Gauss-Seidel method represented by the algorithm

$$\underline{J}^{+(t)}(i,j) = \underline{\underline{R}}^{s}(i,j) \cdot \left[\underline{\underline{T}}_1 \cdot \underline{J}^{+(t-1)}(i+1,j) + \underline{\underline{T}}_2 \cdot \underline{J}^{+(t-1)}(i,j+1) + \underline{\underline{T}}_3 \cdot \underline{J}^{+(t)}(i-1,j) + \right.$$

$$\left. \underline{\underline{T}}_4 \cdot \underline{J}^{+(t)}(i,j-1) \right] + \underline{\underline{R}}^{V}(i,j) \cdot \underline{Q}(i,j), \qquad t = 1, 2, \ldots$$

where (i,j) in (x,y) coordinates refers to the numbering scheme n of the coarse meshes V_n $n = 1,2, \ldots, N$, and the matrices $\underline{\underline{T}}_k$, $k = 1, \ldots, 4$, are composed of permutation matrices.

The eigenvalue problems were solved with source iterations method described in Equation (8), and the standard acceleration schemes were tried in order to accelerate the outer iterations convergence rate. Accelerations by source extrapolation method[10], coarse mesh rebalancing[4] and Chebyshev polynomial method[5] were tried and the best result was obtained with the last method when the overall computational time were reduced by about a factor of 2. The results presented in the next section were obtained with the outer iterations accelerated by Chebyshev polinomial method.

The evaluation of the FERMM for realistic configurations was performed by applying the method to two benchmark problems: the two-dimensional IAEA benchmark problem and the Biblis benchmark problem. The short notation to denote N fine meshes per coarse mesh in the local calculations and M coarse meshes per fuel assembly in the global quadratic/cubic calculations will be described as N/M quadratic/cubic calculations. The processing times are for AMDAHL/V-8 computer at the Computing Center of the University of Michigan.

2-D IAEA Benchmark Problem

For the zone loading PWR core the two-dimensional IAEA benchmark problem were calculated and the results compared with the extrapolated VENTURE calculation[1].

The convergence criterion for the partial current in the inner iterations ε_J is 10^{-2} and for the neutron fluxes in the outer iterations ε_ϕ is about 10^{-5} with maximum of 150 outer iterations and the summary result of the calculation with quadratic and cubic global basis functions is presented in the Table I. One concludes that with an increase in the number of subdomains for response matrix generation the solution converges faster and the results are improved. Better results are also obtained by increasing the number of coarse meshes per fuel assembly but with the penalty of increased computational time due to the slower convergence. The decrease in convergence rate with small mesh sizes is due to the fact that spectral radius of the response matrix increases with decreasing size of coarse meshes[12].

The results are, in general, well within the accuracy expected from a coarse mesh method[3] and at the same time economical in view of the fine-mesh finite defference codes[1].

Table I

Sumary of 2D-IAEA benchmark calculations

(Referer ce (VENTURE) k_{eff} = 1.02969)

Calculation	N° of outer iterations	ϵ_ϕ	CPU (sec)	k_{eff}	maximum /ΔP//P(%)
4/1 quadratic	64	$1.x10^{-5}$	7.1	1.0293	4.25
	51	$1.x10^{-4}$	5.9	1.0293	4.04
9/1 quadratic	66	$1.x10^{-5}$	8.2	1.0296	1.76
	50	$1.x10^{-4}$	6.6	1.0296	1.52
16/1 quadratic	66	$1.x10^{-5}$	10.2	1.0296	.83
	42	$1.x10^{-4}$	7.9	1.0296	.54
9/4 quadratic	150	$3.x10^{-5}$	47.4	1.0295	1.41
	106	$1.x10^{-4}$	34.8	1.0294	1.18
16/4 quadratic	150	$2.x10^{-5}$	49.8	1.0294	.77
	90	$1.x10^{-4}$	32.4	1.0293	.26
4/1 cubic	91	$1.x10^{-5}$	16.3	1.0298	2.04
	72	$1.x10^{-4}$	13.1	1.0298	1.83
9/1 cubic	99	$1.x10^{-5}$	18.3	1.0299	1.35
	64	$1.x10^{-4}$	12.8	1.0199	1.47
16/1 cubic	83	$1.x10^{-5}$	18.0	1.0298	1.16
	59	$1.x10^{-4}$	14.3	1.0298	1.39
9/4 cubic	149	$1.x10^{-5}$	82.9	1.0295	.64
	93	$1.x10^{-4}$	54.3	1.0295	.65
16/4 cubic	150	$1.3x10^{-5}$	86.4	1.0294	.26
	85	$1.x10^{-4}$	53.9	1.0294	.63

Example of the assembly averaged power distribution is presented in Figure 1 for 9/1 cubic calculation. The thermal neutron flux distribution in the core is illustrated in Figure 2 together with the detailed solution obtained with FEMB[1] (second order Lagrange polynomials in rectangular elements of 5 cm x 5 cm in size). The agreement is relatively good considering the mesh size of 20 cm x 20 cm (9/1 cubic approximation) although some flux discontinuities are observed, as the result of formulation of response matrix method itself where only the continuity of partial currents were imposed.

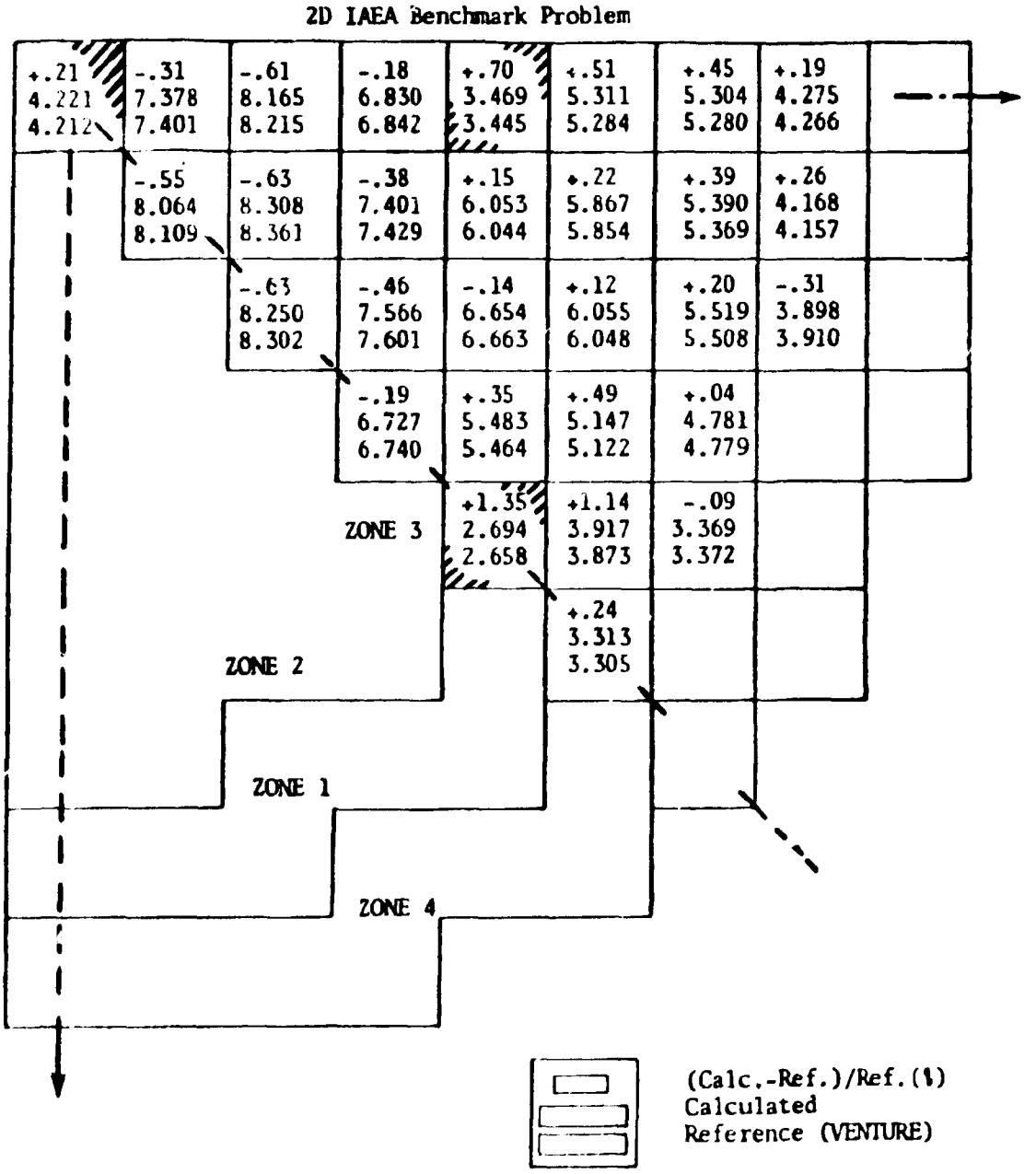


Figure 1 – 2D-IAEA assembly averaged power distribution obtained with 9/1 cubic calculation($\varepsilon\phi = 10^{-5}$).

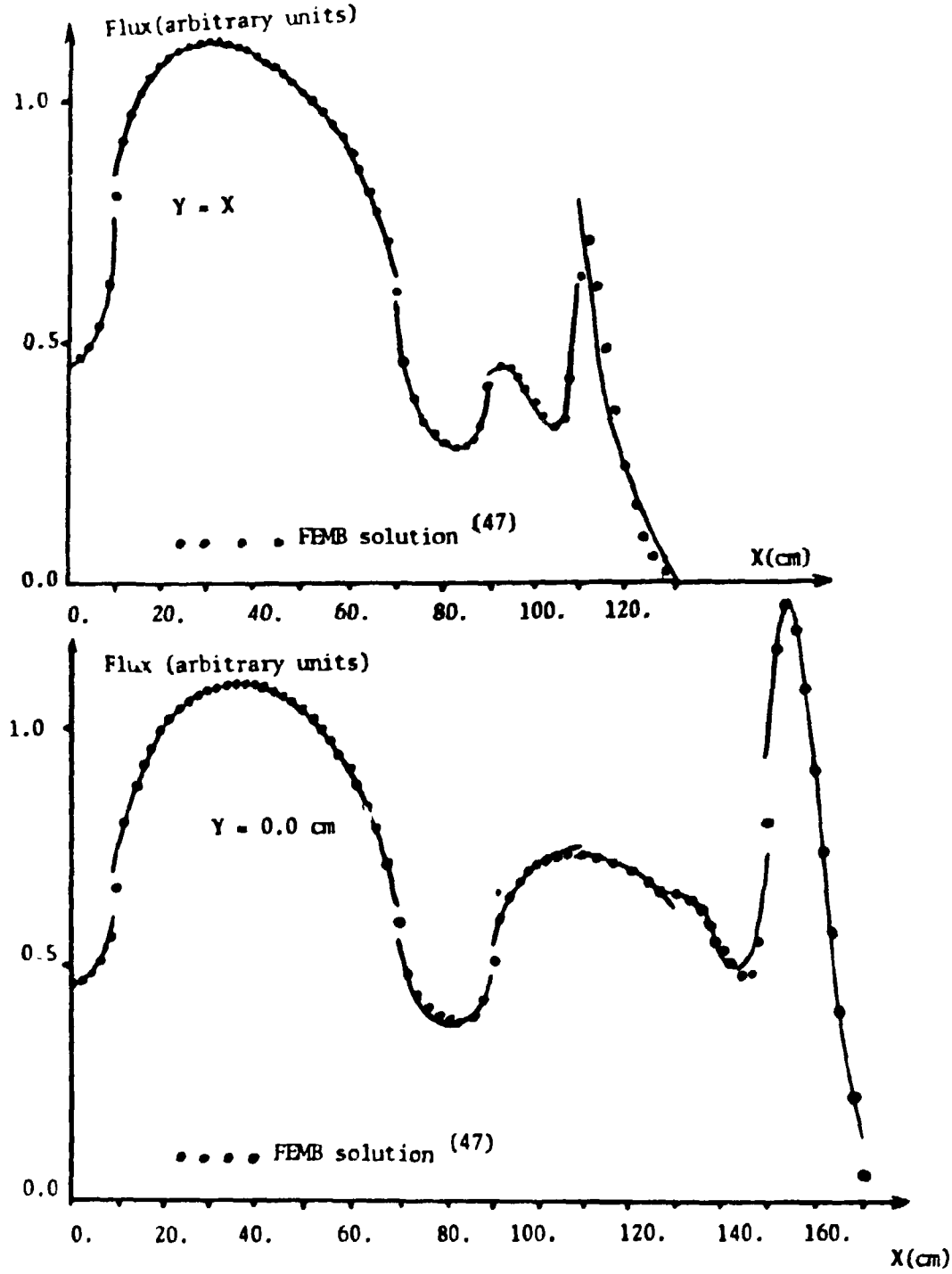


Figure 2 – 2D-IAEA thermal neutron flux distribution obtained with 16/1 cubic calculation ($\varepsilon\phi = 10^{-5}$).

Biblis Benchmark Problem

The finite element response matrix was also tested in a checkerboard loading PWR core, Biblis benchmark problem[11], a highly non-separable (x-y separability) core configuration, and the results compared with the results obtained with the well estabilished coarse mesh NEM code[9] calculation with 5.781 cm x 5.781 cm mesh size. The Figure 3 shows the core configuration with zone assignment and the diffusion parameters are presented in the appendix.

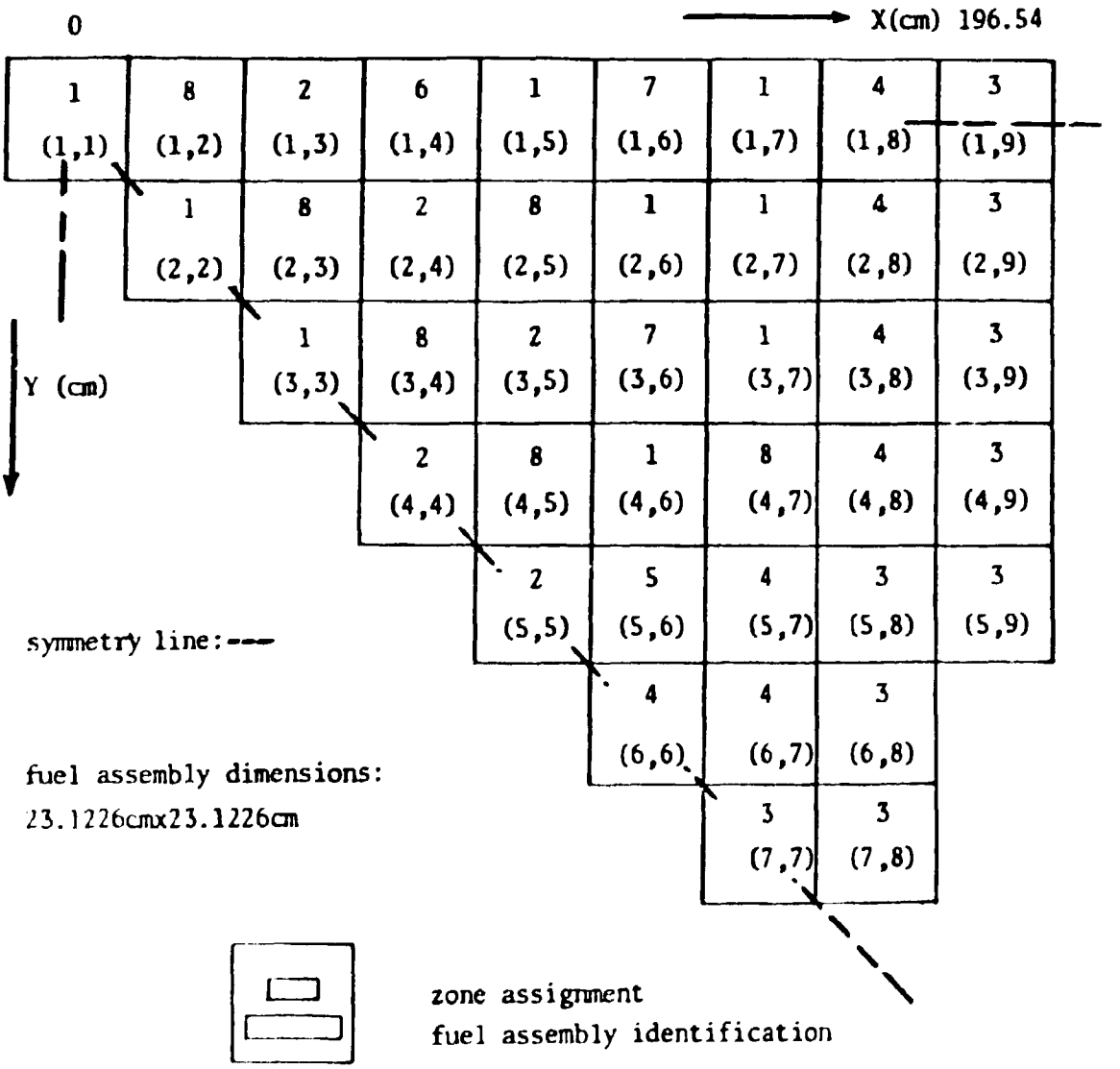


Figure 3 – Biblis benchmark problem [56].

From the summary of Biblis benchmark calculations presented in Table II, one concludes that the 11.561 cm x 11.561 cm coarse mesh calculation, i.e., the 16/4 quadratic and 16/4 cubic calculations, yields essentially the same results as the 5.781 cm x 5.781 cm NEM coarse mesh calculation[11].

It is also noted that 16/1 quadratic and 16/1 cubic calculations yielded results acceptable (within 2% for assembly averaged power levels) for coarse mesh methods but with a significant improvement in computational time, compared with the 16/4 quadratic and 16/4 cubic calculation.

The example of the assembly power distribution and the thermal neutron distribution in the core are illustrated in Figures (4) and (5), respectively.

Concluding Remarks

In the present investigation an alternative formulation of the response matrix method implemented with the finite element method for application to coarse mesh reactor analysis has been developed.

Table II

Sumary of Biblis benchmark calculations

(Reference (NEM) k_{eff} = 1.02511)

Calculation	N° of outer iterations	ϵ_ϕ	CPU (sec)	k_{eff}	maximum /ΔP//P(%)
9/1 quadratic	73	$1x10^{-4}$	9.9	1.0255	2.79
	99	$1x10^{-5}$	12.5	1.0255	2.51
16/1 quadratic	66	$1x10^{-4}$	13.6	1.0253	1.72
	99	$3x10^{-5}$	16.7	1.0253	1.17
16/4 quadratic	101	$1x10^{-4}$	41.5	1.0250	.90
	150	$3x10^{-5}$	55.6	1.0250	.20
9/1 cubic	52	$1x10^{-4}$	13.9	1.0255	3.67
	99	$1x10^{-5}$	21.8	1.0255	2.96
16/1 cubic	54	$1x10^{-4}$	19.9	1.0254	2.11
	99	$2x10^{-5}$	26.9	1.0254	1.55
16/4 cubic	102	$1x10^{-4}$	70.5	1.0250	.87
	150	$2x10^{-5}$	96.8	1.0250	.23

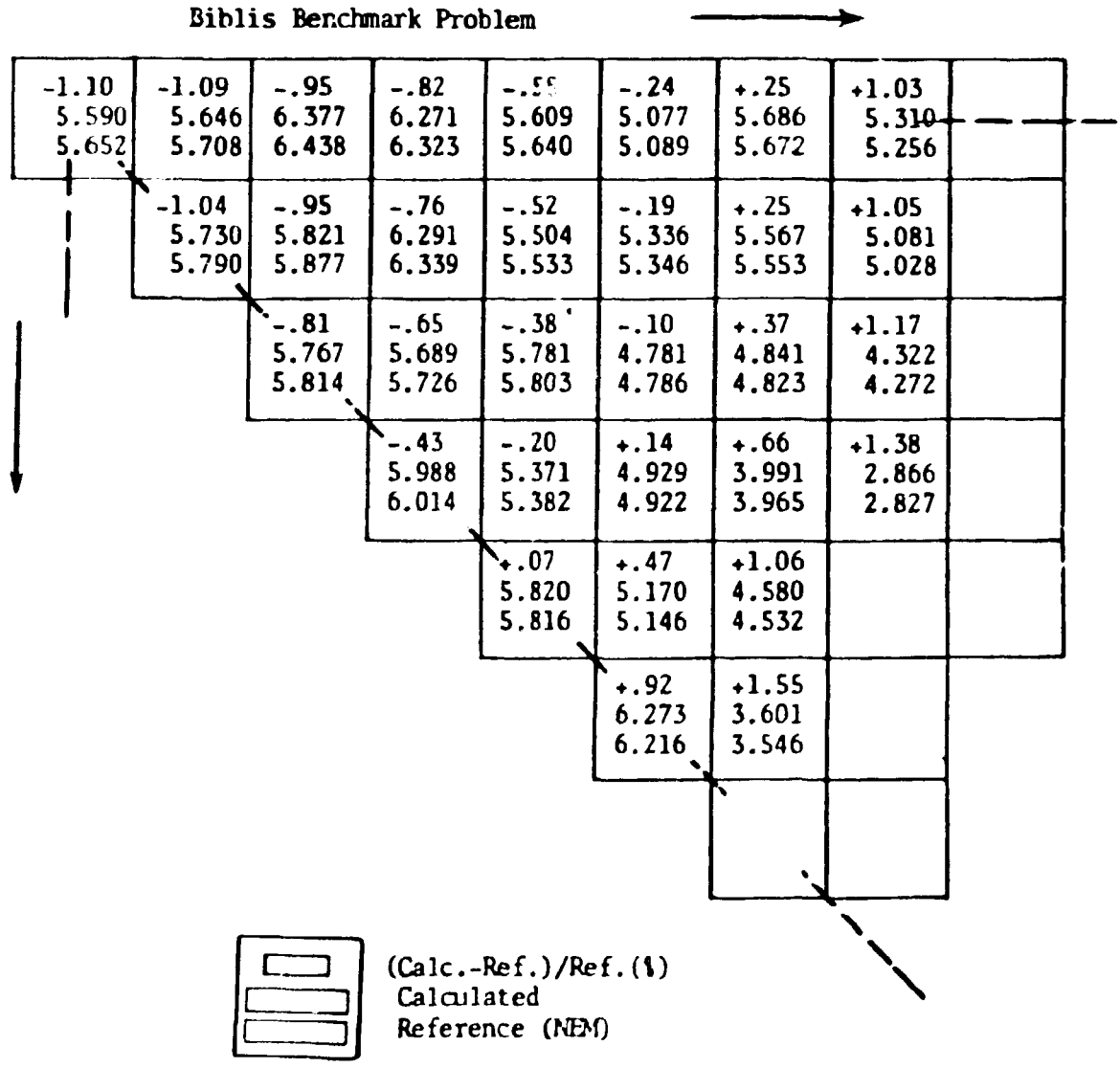
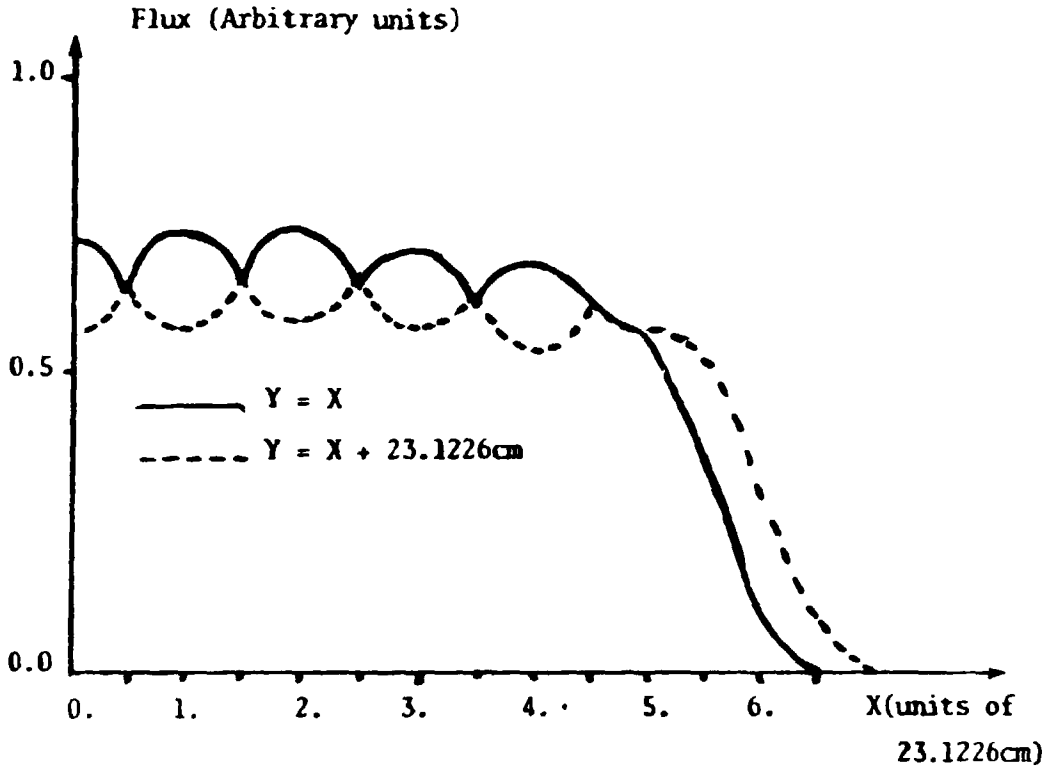


Figure 4 – Biblis assembly averaged power distribution obtained with 16/1 cubic calculation ($\varepsilon\phi = 2.\times10^{-5}$).



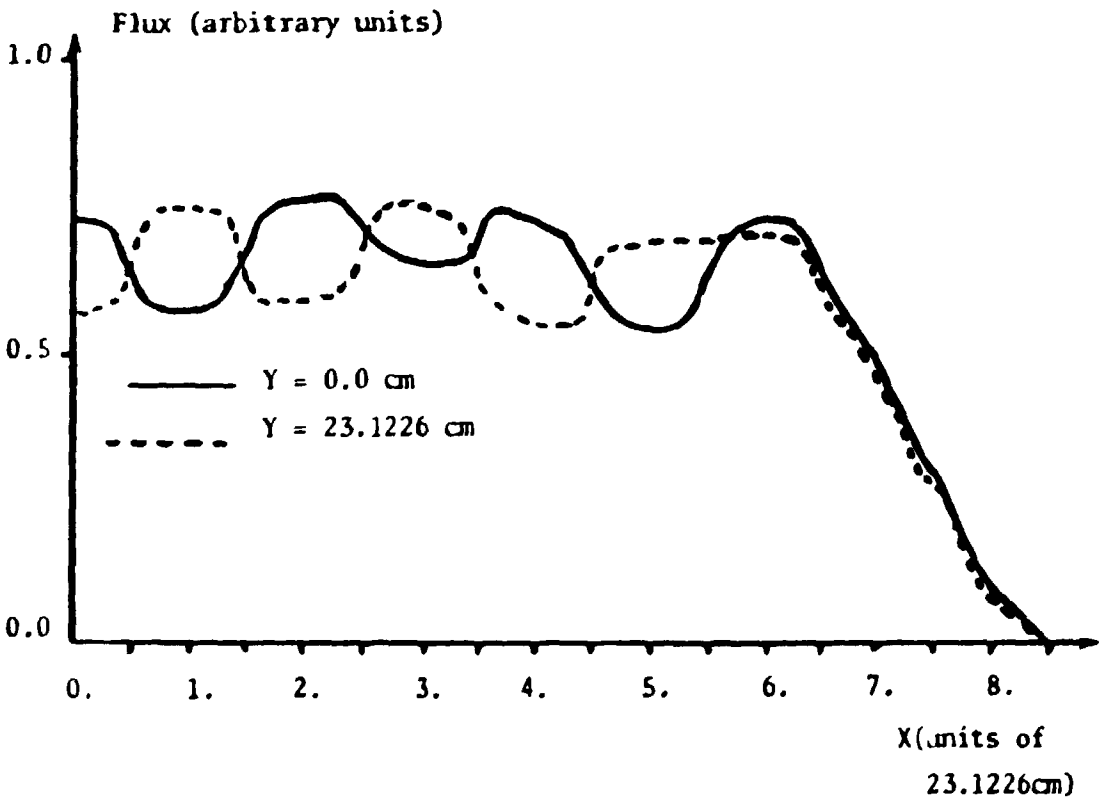


Figure 5 – Biblis thermal neutron distribution obtained with 16/4 cubic calculation ($\epsilon\phi = 2.x10^{-5}$).

The present formulation of the response matrix method, utilizing two types of response matrices, avoided the expense of recalculating the response matrices as a function of keff in the eigenvalue problems. These response matrices, one for the partial current and another for the neutron source, are calculated only at the beginning of the eigenvalue problem, and the outer iterations are performed using conventional solution technique. They are generated with finite element method embedded within the response matrix scheme with a two level structure, in the lower level (assembly or local) the calculation of the local response matrices for each assembly using the weak form of the diffusion equation subject to an arbitrary current and source is performed. The upper level (global or core) calculation is the response matrix calculation for the global partial currents and sources. In the local level two sets of basis functions are defined, one for the interior of the assembly and another for the boundary, and for the global level two additional sets are defined. These definitions allowed convenient calculation of separate response matrices and in addition the treatment of any arbitrary heterogeneous assemblies is possible with the availability of local fluxes.

The present method was applied to two idealized PWR cores, a zone loading 2D-IAEA benchmark problem and a checkerboard loading Biblis benchmark problem, with very good results. The error in the assembly averaged power distribution were within the error observed in other coarse mesh methods[11,9] and the computational time is relatively small compared with the conventional fine-mesh production codes[1].

Furthermore, the unquestionable potential to incorporate additional capabilities such as treatment of spatially dependent cross sections, irregular geometries, and highly heterogeneous fuel assemblies, makes the present technique very attractive for further developments.

APENDIX

Two-Group Diffusion Constants for Biblis Benchmark Problem.

$(B_z^2 = 0.0\ ,\ X^1 = 1.0\ ,\ X^2 = 0.0)$

ZONE	GROUP	D^g	Σ_a^g	Σ_f^g	Σ_f^g	Σ_{12}
1	1	1.4360	.0095042	.0023768	.0058708	.017754
	2	.3635	.0750058	.0388940	.0960670	
2	1	1.4366	.0096785	.0025064	.0061908	.017621
	2	.3636	.0784360	.0419350	.1035800	
3	1	1.3200	.0026562	.0	.0	.023106
	2	.2772	.0715960	.0	.0	
4	1	1.4389	.0103630	.0030173	.0074527	.017101
	2	.3638	.0914080	.0535870	.1323600	
5	1	1.4381	.0100030	.0025064	.0061908	.017290
	2	.3665	.0848280	.0419350	.1035800	
6	1	1.4385	.0101320	.0026026	.0064285	.017192
	2	.3665	.0873140	.0441740	.1091100	
7	1	1.4389	.0101650	.0025064	.0061908	.017125
	2	.3679	.0880240	.0419350	.1035800	
8	1	1.4393	.0102940	.0026026	.0064285	.017027
	2	.3680	.0905100	.0441740	.1091100	

RESUMO

Desenvolveu-se uma nova técnica baseada em formulação alternativa do método de matriz resposta implementada com a técnica de elementos finitos. Dois tipos de matriz resposta são gerados pela técnica de resíduos ponderados de Galerkin aplicada para resolver a forma fraca da equação de difusão sujeita à condição de contorno específica. Os polinômios são definidos em domínios discretos e em dois níveis: o primeiro para cálculos locais (nível de elementos de combustível) e o segundo para cálculos globais (nível de caroço do reator). A presente técnica foi testada em dois problemas bi-dimensionais, reator de teste 2D-IAEA e Biblis, obtendo-se resultados satisfatórios. O tempo computacional é da mesma ordem de grandeza dos programas de malhas grossas atualmente em escala de produção. Ademais a utilização da técnica de elementos finitos para a solução da formulação modificada do método de matriz resposta permite que o método seja capaz de tratar as seções de choque espacialmente variáveis dentro de uma malha grossa. Há também facilidade de se tratar diferentes configurações geométricas e elementos de combustível com alta heterogeneidade local.

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