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THE F_N METHOD FOR SOLVING THE CRITICAL PROBLEM FOR A SLAB WITH A FINITE REFLECTOR

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Abstract—The F_N method is used to solve the critical problem for a slab reactor with a finite reflector. Numerical results for the critical thickness are shown for different values of the mean number of secondary neutrons per collision and reflector thickness using various orders of the F_N approximation. The results show the capability of the method to provide accurate solutions.

1. INTRODUCTION

The critical problem for a slab reactor with infinite reflector has been solved by Siewert and Burkart[1] through the use of a combination of Case's technique[2] with Chandrasekhar's invariance principles.[3] The case of the finite reflector was solved by Neshat, Siewert and Ishiguro[4] using the principles of invariance and the P_N method; Ishiguro and Garcia[5] also solved this problem using a new method of regularization. Recently,[6] this problem was reviewed and equations for the expansion coefficients were summarized.

Here, we wish to apply the recently developed F_N method[7,8] to the above problem in order to demonstrate the simplicity of the method and the computation merits of the technique.

We consider the one-speed transport equations for the core, $-\tau \le x \le \tau$, and the reflector, $\tau < |x| < b$, written in the familiar manner[2]

$$\mu \frac{\partial}{\partial x} \psi_\alpha(x,\mu) + \psi_\alpha(x,\mu) = \tfrac{1}{2} c_\alpha \int_{-1}^{1} \psi_\alpha(x,\mu')\,\mathrm{d}\mu', \qquad \alpha = 1 \text{ and } 2, \tag{1}$$

where $\alpha = 1$ implies the core and $\alpha = 2$ the reflector. We take $c_1 > 1$ and $c_2 \le 1$ and seek solutions of equation (1) subject to the boundary conditions

$$\psi_1(\tau,\mu) = \psi_2(\tau,\mu), \quad \mu > 0, \quad \mu < 0, \tag{2a}$$

$$\psi_2(b,-\mu) = 0, \quad \mu > 0, \tag{2b}$$

and

$$\psi_2(-b,\mu) = 0, \quad \mu > 0. \tag{2c}$$

Here we consider c_1 and c_2, as well as the reflector thickness, $\Delta = b - \tau$ to be given and thus seek the critical half-thickness τ.

2. ANALYSIS

The solution of equations (1) can be written in terms of the elementary solutions[2] as

$$\psi_1(x,\mu) = A(\nu_0)[\phi_1(\nu_0,\mu)e^{-x/\nu_0} + \phi_1(-\nu_0,\mu)e^{x/\nu_0}] + \int_0^1 A(\nu)[\phi_1(\nu,\mu)e^{-x/\nu} + \phi_1(-\nu,\mu)e^{x/\nu}]\,\mathrm{d}\nu \tag{3}$$

and

$$\begin{aligned}\psi_2(x,\mu) = {} & B(\eta_0)\phi_2(\eta_0,\mu)e^{-x/\eta_0} \\ & + B(-\eta_0)\phi_2(-\eta_0,\mu)e^{x/\eta_0} \\ & + \int_0^1 B(\eta)\phi_2(\eta,\mu)e^{-x/\eta}\,\mathrm{d}\eta \\ & + \int_0^1 B(-\eta)\phi_2(-\eta,\mu)e^{x/\eta}\,\mathrm{d}\eta.\end{aligned} \tag{4}$$

Using the full-range orthogonality properties of the eigenfunctions, we note that

$$\int_{-1}^{1} \mu\phi_1(-\xi,\mu)\psi_1(x,\mu)\,\mathrm{d}\mu = N_1(-\xi)A(-\xi)e^{x/\xi}, \qquad \xi \in P_1, \tag{5a}$$

and

$$\int_{-1}^{1} \mu\phi_2(-\hat{\xi},\mu)\psi_2(x,\mu)\,\mathrm{d}\mu = N_2(-\hat{\xi})B(-\hat{\xi})e^{x/\hat{\xi}}, \qquad \hat{\xi} \in P_2, \tag{5b}$$

where $N_\alpha(-\xi)$ are the full-range normalization factor[2] and P_1 and P_2 are the intervals defined by $P_1 = \nu_0 \cup (0,1)$ and $P_2 = \eta_0 \cup (0,1)$.

If we write equation (5b) for $x = \tau$ and $x = b$, and then eliminate $N_2(-\hat{\xi})B(-\hat{\xi})$ between the resulting

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equations we obtain

$$\int_0^1 \mu\phi_2(-\hat{\xi},\mu)\psi_2(\tau,\mu)\,d\mu - \int_0^1 \mu\phi_2(\hat{\xi},\mu)\psi_2(\tau,-\mu)\,d\mu = e^{-\Delta/\hat{\xi}}\int_0^1 \mu\phi_2(-\hat{\xi},\mu)\psi_2(b,\mu)\,d\mu, \quad (6)$$

and writing equation (5b) for $x = -\tau$ and $x = -b$, we find

$$-\int_0^1 \mu\phi_2(\hat{\xi},\mu)\psi_2(-b,-\mu)\,d\mu = e^{-\Delta/\hat{\xi}}\left[\int_0^1 \mu\phi_2(-\hat{\xi},\mu)\psi_2(-\tau,\mu)\,d\mu - \int_0^1 \mu\phi_2(\hat{\xi},\mu)\psi_2(-\tau,-\mu)\,d\mu\right]. \quad (7)$$

In the same manner, we write equation (5a) for $x = \tau$ and $x = -\tau$, to obtain

$$\int_0^1 \mu\phi_1(-\xi,\mu)\psi_1(-\tau,\mu)\,d\mu - \int_0^1 \mu\phi_1(\xi,\mu)\psi_1(-\tau,-\mu)\,d\mu = e^{-2\tau/\xi}\left[\int_0^1 \mu\phi_1(-\xi,\mu)\psi_1(\tau,\mu)\,d\mu - \int_0^1 \mu\phi_1(\xi,\mu)\psi_1(\tau,-\mu)\,d\mu\right]. \quad (8)$$

Now, we can introduce the F_N approximation, as reported by Grandjean and Siewert,[8]

$$\psi_1(\tau,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha, \qquad \mu > 0, \quad (9a)$$

$$\psi_1(\tau,-\mu) = \sum_{\alpha=0}^{N} b_\alpha \mu^\alpha, \qquad \mu > 0, \quad (9b)$$

and

$$\psi_2(b,\mu) = \psi_2(-b,-\mu) = \sum_{\alpha=0}^{N} e_\alpha \mu^\alpha, \quad \mu > 0, \quad (9c)$$

into equations (6), (7), and (8). If we make use of the boundary conditions, equations (2), we obtain the F_N equations for this problem, i.e.

$$\sum_{\alpha=0}^{N} a_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} b_\alpha B_\alpha^{(2)}(\hat{\xi}) = e^{-\Delta/\hat{\xi}}\sum_{\alpha=0}^{N} e_\alpha A_\alpha(\hat{\xi}), \quad (10)$$

$$\sum_{\alpha=0}^{N} b_\alpha A_\alpha(\hat{\xi}) - \sum_{\alpha=0}^{N} a_\alpha B_\alpha^{(2)}(\hat{\xi}) = -e^{\Delta/\hat{\xi}}\sum_{\alpha=0}^{N} e_\alpha B_\alpha^{(2)}(\hat{\xi}) \quad (11)$$

and

$$\sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi) - \sum_{\alpha=0}^{N} a_\alpha B_\alpha^{(1)}(\xi) = e^{-2\tau/\xi}\left[\sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi) - \sum_{\alpha=0}^{N} b_\alpha B_\alpha^{(1)}(\xi)\right] \quad (12)$$

where $A_\alpha(\xi)$ and $B_\alpha^{(i)}(\xi)$ are those quantities given by Grandjean and Siewert:[8]

$$B_\alpha^{(i)}(\xi) = \xi B_{\alpha-1}^{(i)}(\xi) - \frac{1}{\alpha+1}, \quad \alpha \geq 1,\ i = 1,2, \quad (13a)$$

with

$$B_0^{(i)}(\xi) = \frac{2}{c_i} - 1 - \xi\log\left(1 + \frac{1}{\xi}\right), \quad i = 1,2, \quad (13b)$$

and

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha+1}, \quad \alpha \geq 1, \quad (14a)$$

with

$$A_0(\xi) = 1 - \xi\log\left(1 + \frac{1}{\xi}\right). \quad (14b)$$

To find the critical half-thickness we let $a_0 = 1$ and give as an initial value τ as computed by the F_0 approximation. We then solve equations (10), (11) with $\hat{\xi} \in P_2$ and equation (12) with $\xi \in (0,1)$ for the coefficients a_α, b_α, and e_α. Finally we insert these coefficients into

$$e^{-2\tau/\nu_0}\sum_{\alpha=0}^{N}[b_\alpha B_\alpha^{(1)}(\nu_0) - a_\alpha A_\alpha(\nu_0)] = \sum_{\alpha=0}^{N}[a_\alpha B_\alpha^{(1)}(\nu_0) - b_\alpha A_\alpha(\nu_0)] \quad (15)$$

in order to find a new value for τ. The iteration is repeated until a converged result is obtained. As an example, we take the initial value of τ as computed by F_0 approximation:

$$\tau^{(0)} = -\frac{\nu_0}{2}\log\left[\frac{b_0 A_0(\nu_0) - B_0^{(1)}(\nu_0)}{A_0(\nu_0) - b_0 B_0^{(1)}(\nu_0)}\right] \quad (16)$$

with

$$b_0 = \frac{A_0(\eta_0)B_0^{(2)}(\eta_0)(1 - e^{-2\Delta/\eta_0})}{[B_0^{(2)}(\eta_0)]^2 - [A_0(\eta_0)]^2 e^{-2\Delta/\eta_0}}. \quad (17)$$

3. NUMERICAL RESULTS

To solve equations (10), (11), and (12) for the coefficients, we pick values of $\hat{\xi} \in P_2$ and $\xi \in (0,1)$ according to the scheme suggested by Grandjean and

Table 1. Cases studied

Case	c_1	c_2	$\Delta = b - \tau$
1	1.01	0.09	0.5
2	1.01	0.90	1.0
3	1.30	0.09	0.5
4	1.30	0.90	1.0
5	1.50	0.09	0.5
6	1.50	0.90	1.0
7	1.91	0.09	0.5
8	1.91	0.90	1.0

Siewert;[8] i.e. for $\hat{\xi}$ we pick $\hat{\xi}_0 = \eta_0$, $\hat{\xi}_1 = 0$, $\hat{\xi}_2 = 1$, and the remaining $\hat{\xi}_\beta$ spaced equally in $[0, 1]$, and for ξ we pick $\xi_0 = 0$, $\xi_1 = 1$, and the remaining also equally spaced in $[0, 1]$. For each iteration we solve $(3N + 2)$ linear system of equations. The scheme generally converges in three or four iterations.

In Table 1, we show the cases studied, and in Table 2, we report the values of the half-thickness for various orders of approximation, and the 'exact' value reported by Burkart.[9,10]

4. CONCLUSIONS

We note that the F_N method yields results that are accurate to three or four significant figures for $N \leq 5$, which is remarkably good considering the simplicity of the method. The computation time on the IBM 370/165 machine is on the order of 3 s per case for $N \leq 8$. In short, we conclude that: (i) the F_N method is a remarkably simple and easy to use method; (ii) the method yields accurate results; and (iii) the computation time is very short.

We hope soon to be able to apply the method to the more general problem of a slab reactor with a blanket and a reflector.

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Table 2. Critical half-thickness

Case	F_0	F_3	F_4	F_5	F_6	F_7	Exact
1	8.3587	8.3106	8.3106	8.3107	8.3107	8.3107	8.3107
2	7.7346	7.6776	7.6777	7.6777	7.6778	7.6778	7.6778
3	0.9323	0.9246	0.9245	0.9245	0.9245	0.9246	0.9246
4	0.6230	0.6025	0.6026	0.6026	0.6026	0.6027	0.6027
5	0.5891	0.5939	0.5943	0.5943	0.5943	0.5943	0.5943
6	0.3671	0.3590	0.3594	0.3597	0.3597	0.3597	0.3597
7	0.3171	0.3324	0.3334	0.3343	0.3346	0.3346	0.3346
8	0.1841	0.1871	0.1883	0.1886	0.1892	0.1893	0.1893