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ANISOTROPIC SCATTERING: HALF-RANGE ORTHOGONALITY
AND CRITICAL SLAB PROBLEM**

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TWO-GROUP NEUTRON-TRANSPORT THEORY WITH LINEARLY ANISOTROPIC SCATTERING: HALF-RANGE ORTHOGONALITY AND CRITICAL SLAB PROBLEM

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ABSTRACT

Half-range adjoint functions concerning the elementary solutions of the two-group one-dimensional neutron-transport equation for linearly anisotropic scattering are established. The half-range orthogonality relation is based on matrix functions which can be computed from regular Fredholm-type equations. All necessary integrals are evaluated and several typical problems are solved in a concise manner.

The established orthogonality relation is used to reduce critical slab problems to a suitable computational form. Exact values of critical dimensions are reported for selected sets of parameters. Computations of the eigenvalues and fundamental matrices are also discussed.

PART I - HALF-RANGE ORTHOGONALITY

1. Introduction

Since the introduction of the singular eigenfunction-expansion technique [1] rigorous solutions of the transport equations have been obtained for various problems. The earliest applications of the Case method centered around the one-speed model and full-range problems in plane geometry. Though various problems of theoretical and practical interest can be defined by full-range boundary conditions, half-range problems have more physical interest. Kuscer et al. [4] were the first to observe half-range orthogonality properties of the elementary solutions of the one-speed transport equation in plane geometry and showed that half-range problems could be solved readily in a straightforward manner.

Applications of the Case method to the multigroup transport equation have also been successful in both neutron transport theory and radiative transfer. Half-range orthogonality properties of the elementary solutions have also been shown for several cases of the two-group model. Siewert and Zweitel [10], considering a special case of the two-group model, were able to express the half-range weight matrix in terms of a scalar function.

More recently, Siewert [8] obtained half-range orthogonality relations for Chandrasekhar's model for the scattering of polarized light in terms of a Chandrasekhar-type H matrix. Similar half-range results have been reported by Siewert and Ishiguro [9] for the general two-group neutron-transport equation for isotropic scattering and it has been shown that the obtained half-range orthogonality relation can be used to solve various half-space problems in a concise manner. Further, the half-range orthogonality relation was used by Kriese et al. [3] to reduce two-group critical problems for slabs and spheres to a suitable computational form.

In recent years there has been considerable interest in extending the analysis of multigroup transport problems to include the effects of anisotropic scattering. Reith and Siewert [6] proved the full-range completeness and orthogonality properties of the elementary

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solutions of the two-group neutron-transport equation for linearly anisotropic scattering. We would like to report here half-range orthogonality relations for the model considered by Reith and Siewert [6]. We thus consider the equation

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\Psi}(x, \mu) + \underset{\sim}{\Sigma} \underset{\sim}{\Psi}(x, \mu) = \underset{\sim}{C} \int_{-1}^1 \underset{\sim}{\Psi}(x, \mu') d\mu' + \mu \underset{\sim}{B} \int_{-1}^1 \underset{\sim}{\Psi}(x, \mu') \mu' d\mu' \quad (1)$$

where the elements of the two-vector $\underset{\sim}{\Psi}(x, \mu)$ are the group angular fluxes, μ is the direction cosine of the neutrons (as measured from the positive x-axis) and x is the optical variable defined (without loss of generality) in terms of the smaller of the two total group cross-sections. With the optical variable so defined, the $\underset{\sim}{\Sigma}$ -matrix is given by

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma > 1, \quad (2)$$

where σ denotes the ratio of the two total cross-sections. In addition, the elements of the arbitrary real matrices $\underset{\sim}{C}$ and $\underset{\sim}{B}$ are denoted by $c_{\alpha\beta}$ and $b_{\alpha\beta}$, $\alpha, \beta = 1$ and 2 , respectively.

Although Reith and Siewert [6] have reported the elementary solutions of Equation (1) and the full-range completeness and orthogonality properties of the solutions, we use a slightly different notation. We would, therefore, like to summarize our results here in order to establish the required formalism. We rewrite Equation (1) as

$$\mu \frac{\partial}{\partial x} \underset{\sim}{\Psi}(x, \mu) + \underset{\sim}{\Sigma} \underset{\sim}{\Psi}(x, \mu) = \underset{\sim}{Q}(\mu) \underset{\sim}{D} \int_{-1}^1 \underset{\sim}{Q}(\mu') \underset{\sim}{\Psi}(x, \mu') d\mu', \quad (3)$$

where we have defined a 2×4 matrix $\underset{\sim}{Q}(\mu)$ and a 4×4 matrix $\underset{\sim}{D}$ as

$$\underset{\sim}{Q}(\mu) = \begin{bmatrix} \underset{\sim}{I} & \mu \underset{\sim}{I} \end{bmatrix}, \quad \underset{\sim}{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (4)$$

and

$$\underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{C} & \underset{\sim}{Q} \\ \underset{\sim}{Q} & \underset{\sim}{B} \end{bmatrix} \quad (5)$$

Seeking solutions of Equation (3) of the form:

$$\underset{\sim}{\Psi}_{x\xi}(x, \mu) = \underset{\sim}{F}(\xi, \mu) e^{-x/\xi} \quad (6)$$

yields

$$\begin{bmatrix} \Sigma & -\frac{\mu}{\xi} \\ \tilde{\Lambda} & \tilde{\Lambda} \end{bmatrix} \tilde{F}(\xi, \mu) = \tilde{Q}(\mu) \tilde{D}\tilde{\Gamma}(\xi) \tilde{M}(\xi), \quad (7)$$

and thus, in general,

$$\tilde{F}(\xi, \mu) = \tilde{E}(\xi, \mu) \tilde{Q}(\mu) \tilde{D}\tilde{\Gamma}(\xi) \tilde{M}(\xi), \quad (8)$$

where

$$\tilde{M}(\xi) = \int_{-1}^1 \tilde{F}(\xi, \mu) d\mu, \quad (9)$$

$$\tilde{\Lambda}(\xi) = \begin{bmatrix} \tilde{\Lambda} & \\ \xi & \left[\tilde{\Sigma} - 2\tilde{C} \right] \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda} \\ \xi \tilde{\gamma} \end{bmatrix}, \quad (10)$$

and

$$\tilde{E}(\xi, \mu) = \begin{bmatrix} \tilde{\Sigma} & -\frac{\mu}{\xi} \\ \tilde{\Lambda} & \tilde{\Lambda} \end{bmatrix}^{-1} \quad (11)$$

Regarding the discrete spectrum $\xi = \pm\nu_i \notin (-1, 1)$, we note that

$$\tilde{F}(\pm\nu_i, \mu) = \tilde{E}(\pm\nu_i, \mu) \tilde{Q}(\mu) \tilde{D}\tilde{\Gamma}(\pm\nu_i) \tilde{M}(\nu_i), \quad (12)$$

where $\pm\nu_i$ are the zeros of the dispersion function $\Lambda(z) = \det \tilde{\Lambda}(z)$ with

$$\Lambda(z) = \tilde{\Lambda} + z \int_{-1}^1 \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu - z} \tilde{D}\tilde{\Gamma}(z), \quad (13)$$

where the superscript * is used to denote an operation to change $\mu \rightarrow \sigma\mu$ in the top row, thus

$$\tilde{Q}^*(\mu) = \begin{bmatrix} \tilde{\Lambda} & \mu \tilde{\Sigma} \\ \tilde{\Lambda} & \tilde{\Lambda} \end{bmatrix}. \quad (14)$$

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In addition

$$\tilde{\Theta}(\mu) = \begin{bmatrix} \theta(\mu) & 0 \\ 0 & 1 \end{bmatrix}, \quad (15)$$

with $\theta(\mu) = 1, \mu \in (-\frac{1}{\sigma}, \frac{1}{\sigma})$, and $\theta(\mu) = 0$, otherwise.

$$\tilde{E}(\pm\nu_i, \mu) = \begin{bmatrix} \frac{\nu_i}{\sigma\nu_i \mp \mu} & 0 \\ 0 & \frac{\nu_i}{\nu_i \mp \mu} \end{bmatrix} \quad (16)$$

and

$$\tilde{\Lambda}(\pm\nu_i) \tilde{M}(\nu_i) = \tilde{Q}. \quad (17)$$

We note that the dispersion function has pairs of zeros $\pm\nu_i$; here ν_i denotes a discrete eigenvalue with a positive real (imaginary) part. We use κ to denote the number of pairs of discrete eigenvalues. In the limit $|z| \rightarrow \infty$, Equation (13) yields

$$\tilde{\Lambda}(\infty) = \tilde{I} - 2\tilde{\Sigma}^{-1} \left[\tilde{C} + \frac{1}{3} \tilde{\Sigma}^{-1} \tilde{A} \right], \quad (18)$$

where

$$\tilde{A} = \tilde{B}\tilde{\chi}. \quad (19)$$

We assume here that $\tilde{\Lambda}(\infty) \neq 0$.

For the continuum, $\xi = \nu \in (-1, 1)$, we write the solution to Equation (7) as

$$\tilde{F}(\nu, \mu) = \left[\nu \tilde{K}(\nu, \mu) + \omega(\nu) \tilde{\delta}(\nu, \mu) \right] \tilde{Q}(\mu) \tilde{D} \tilde{\Gamma}(\nu) \tilde{M}(\nu), \quad (20)$$

where

$$\tilde{K}(\nu, \mu) = \begin{bmatrix} \text{Pv}(\frac{1}{\sigma\nu - \mu}) & 0 \\ 0 & \text{Pv}(\frac{1}{\nu - \mu}) \end{bmatrix} \quad \text{and} \quad \tilde{\delta}(\nu, \mu) = \begin{bmatrix} \delta(\sigma\nu - \mu) & 0 \\ 0 & \delta(\nu - \mu) \end{bmatrix}. \quad (21)$$

Here the distribution $\text{Pv}(\frac{1}{x})$ denotes that ensuing integrals are to be evaluated in the Cauchy principal-value sense, and $\delta(x)$ is the Dirac delta distribution. Equation (20) when integrated over μ from -1 to 1 yields

$$\left[\underset{\sim}{\lambda}(\nu) - \omega(\nu) \underset{\sim}{\Theta}(\nu) \underset{\sim}{Q}^*(\nu) \underset{\sim}{D}\Gamma(\nu) \right] \underset{\sim}{M}(\nu) = \underset{\sim}{0} \quad (22)$$

and therefore the parameter $\omega(\nu)$ is to be determined from the requirement

$$\det \left[\underset{\sim}{\lambda}(\nu) - \omega(\nu) \underset{\sim}{\Theta}(\nu) \underset{\sim}{Q}^*(\nu) \underset{\sim}{D}\Gamma(\nu) \right] = \underset{\sim}{0} \quad (23)$$

where

$$\underset{\sim}{\lambda}(\nu) = \underset{\sim}{1} + \nu \text{P} \int_{-1}^1 \underset{\sim}{\Theta}(\mu) \underset{\sim}{Q}^*(\mu) \frac{\omega\mu}{\mu-\nu} \underset{\sim}{D}\Gamma(\nu) . \quad (24)$$

If we now let

$$\text{Region } \textcircled{1} \Rightarrow \nu \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right)$$

$$\text{Region } \textcircled{2} \Rightarrow \nu \in \left(-1, -\frac{1}{\sigma}\right) \cup \left(-\frac{1}{\sigma}, 1\right) ,$$

then Equation (23) yields two solutions $\omega_1^{(1)}(\nu)$ and $\omega_2^{(1)}(\nu)$ for $\nu \in \textcircled{1}$ and one solution $\omega^{(2)}(\nu)$ for $\nu \in \textcircled{2}$. We thus find

$$\begin{aligned} \underset{\sim}{F}_{\underset{\sim}{\alpha}}^{(1)}(\nu, \mu) &= \underset{\sim}{E}_{\underset{\sim}{\alpha}}^{(1)}(\nu, \mu) \underset{\sim}{Q}(\mu) \underset{\sim}{D}\Gamma(\nu) \underset{\sim}{M}_{\underset{\sim}{\alpha}}^{(1)}(\nu) \\ &= \left[\nu \underset{\sim}{K}(\nu, \mu) + \omega_{\underset{\sim}{\alpha}}^{(1)}(\nu) \underset{\sim}{\delta}(\nu, \mu) \right] \underset{\sim}{Q}(\mu) \underset{\sim}{D}\Gamma(\nu) \underset{\sim}{M}_{\underset{\sim}{\alpha}}^{(1)}(\nu) , \\ \nu &\in \textcircled{1}, \quad \alpha = 1 \text{ and } 2, \end{aligned} \quad (25)$$

and

$$\begin{aligned} \underset{\sim}{F}_{\underset{\sim}{\alpha}}^{(2)}(\nu, \mu) &= \underset{\sim}{E}_{\underset{\sim}{\alpha}}^{(2)}(\nu, \mu) \underset{\sim}{Q}(\mu) \underset{\sim}{D}\Gamma(\nu) \underset{\sim}{M}_{\underset{\sim}{\alpha}}^{(2)}(\nu) \\ &= \left[\nu \underset{\sim}{K}(\nu, \mu) + \omega^{(2)}(\nu) \underset{\sim}{\delta}(\nu, \mu) \right] \underset{\sim}{Q}(\mu) \underset{\sim}{D}\Gamma(\nu) \underset{\sim}{M}_{\underset{\sim}{\alpha}}^{(2)}(\nu) , \\ \nu &\in \textcircled{2} . \end{aligned} \quad (26)$$

Reith and Siewert [6] have shown that the established normal modes form a complete basis set for the full-range expansion of arbitrary two-vector Hölder functions. Thus we note that our general solution to Equation (1) can be written as

$$\begin{aligned} \tilde{\Psi}(x, \mu) = & \sum_{i=1}^{\kappa} \left[A(\nu_i) F_{\nu_i}^{(1)}(-x/\nu_i) e^{-x/\nu_i} + A(-\nu_i) F_{\nu_i}^{(1)}(x/\nu_i) e^{x/\nu_i} \right] \\ & + \int_{\mathbb{O}} \left[A_1^{(1)}(\nu) F_{\nu}^{(1)}(\nu, \mu) + A_2^{(1)}(\nu) F_{\nu}^{(1)}(\nu, \mu) \right] e^{-x/\nu} d\nu \\ & + \int_{\mathbb{Q}} A^{(2)}(\nu) F_{\nu}^{(2)}(\nu, \mu) e^{-x/\nu} d\nu, \end{aligned} \quad (27)$$

where $A(\pm\nu_i)$, $A_{\alpha}^{(1)}(\nu)$ and $A_{\alpha}^{(2)}(\nu)$ are the expansion coefficients to be determined once a specific problem is considered and the resulting boundary conditions specified. In Section 3, we summarize the explicit forms of the normal modes and all necessary normalization integrals reported by Reith and Siewert [6].

To complete this section we should like to introduce the adjoint equation

$$\begin{aligned} \mu \frac{\partial}{\partial x} \tilde{\Psi}_a(x, \mu) + \tilde{\mathcal{L}} \tilde{\Psi}_a(x, \mu) &= \tilde{\mathcal{L}} \int_{-1}^1 \tilde{\Psi}_a(x, \mu') d\mu' + \mu \tilde{\mathcal{P}} \int_{-1}^1 \tilde{\Psi}_a(x, \mu') \mu' d\mu' \\ &= \mathcal{Q}(\mu) \tilde{\mathcal{P}} \int_{-1}^1 \tilde{\mathcal{Q}}(\mu') \tilde{\Psi}_a(x, \mu') d\mu', \end{aligned} \quad (28)$$

where the superscript tilde is used to denote the transpose operation. Again, proposing solutions of the form

$$\tilde{\Psi}_a(x, \mu) = \tilde{\mathcal{L}}_a(\mu) e^{-x/\mu}, \quad (29)$$

we find a set of solutions $\tilde{\mathcal{L}}_a(\pm\nu_i, \mu)$, $\tilde{\mathcal{L}}_{\alpha}^{(1)}(\nu, \mu)$ and $\tilde{\mathcal{L}}_{\alpha}^{(2)}(\nu, \mu)$ analogous to Equations (12), (25) and (26) with, however, a_{ij} and b_{ij} replaced by a_{ji} and b_{ji} . We note the dispersion matrix for this case is

$$\tilde{\mathcal{D}}_a(z) = \tilde{\mathcal{L}} + z \int_{-1}^1 \tilde{\mathcal{Q}}(\mu) \mathcal{Q}^*(\mu) \frac{d\mu}{\mu - z} \tilde{\mathcal{P}} \tilde{\mathcal{L}}_a(z). \quad (30)$$

where

$$\tilde{\Gamma}_a(z) = \begin{bmatrix} \tilde{1} \\ z\tilde{\gamma} \end{bmatrix}. \quad (31)$$

We also define [6]

$$\tilde{A}_a = \tilde{B}\tilde{\gamma}. \quad (32)$$

Silvernoinen and Zweifel [11] have shown that the direct and adjoint spectra are identical. Thus,

$$\Lambda_a(z) = \det \tilde{\Lambda}_a(z) = \Lambda(z) = \det \tilde{\Lambda}(z). \quad (33)$$

In addition, Reith and Siewert [6] have established the following relationship between the direct and adjoint dispersion matrices:

$$\tilde{\Gamma}(z) \tilde{D} \tilde{Q}^*(z) \tilde{\Lambda}_a(z) = \tilde{\Lambda}(z) \tilde{Q}^*(z) \tilde{D} \tilde{\Gamma}_a(z). \quad (34)$$

2 Half-Range Orthogonality

We consider here a typical half-space problem where we seek a solution to a boundary value constraint of the form

$$\begin{aligned} \tilde{J}(\mu) = & \sum_{i=1}^K A(\nu_i) \tilde{F}(\nu_i, \mu) + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \tilde{F}_1^{(1)}(\nu) + A_2^{(1)}(\nu) \tilde{F}_2^{(1)}(\nu, \mu) \right] d\nu \\ & + \int_{1/\sigma}^1 A^{(2)}(\nu) \tilde{F}^{(2)}(\nu, \mu) d\nu, \quad \mu \in (0, 1). \end{aligned} \quad (35)$$

Here $\tilde{J}(\mu)$ is the expansion function and is considered to be given: it may contain a diverging (at infinity) solution of the transport equation, as for the Milne problem, a particular solution relevant to an inhomogeneous source term or a specified incident distribution. We now wish to establish an orthogonality relation that will allow the solution to Equation (35) to be written in a concise manner;

$$A(\nu_i) = \frac{1}{N(\nu_i)} \left[\tilde{G}(\nu_i, \mu), \tilde{J}(\mu) \right], \quad (36a)$$

$$A_\alpha^{(1)}(\nu) = \frac{1}{N^{(1)}(\nu)} \left[\tilde{G}_\alpha^{(1)}(\nu, \mu), \tilde{J}(\mu) \right], \quad \nu \in (0, \frac{1}{\sigma}), \quad \alpha = 1, 2, \quad (36b)$$

and

$$A^{(2)}(\nu) = \frac{1}{N^{(2)}(\nu)} \left[\underset{\sim}{G}^{(2)}(\nu, \mu), \underset{\sim}{I}(\mu) \right], \quad \nu \in \left(\frac{1}{\sigma}, 1 \right), \quad (36c)$$

where $[X, Y]$ is an appropriate inner product, $N(\nu)$ and $N^{(\alpha)}(\nu)$ are the normalization factors and $\underset{\sim}{G}(\xi, \mu)$ is an appropriate adjoint vector.

We follow invariance principles of Chandrasekhar [2] and thus define the $\underset{\sim}{s}$ -matrix by

$$\underset{\sim}{\Psi}(x, -\mu) = \frac{1}{2\mu} \int_0^1 \underset{\sim}{s}(\mu, \mu') \underset{\sim}{\Psi}(x, \mu') d\mu', \quad \mu \in (0, 1), \quad (37)$$

where $\underset{\sim}{\Psi}(x, \mu)$ is a half-space solution of Equation (1).

In the standard manner [2,5,7] we can derive an equation for the $\underset{\sim}{s}$ -matrix

$$\begin{aligned} & \frac{1}{\mu} \underset{\sim}{\Sigma} \underset{\sim}{s}(\mu, \mu_0) + \frac{1}{\mu_0} \underset{\sim}{s}(\mu, \mu_0) \underset{\sim}{\Sigma} \\ &= 2 \left[\underset{\sim}{Q}(-\mu) + \frac{1}{2} \int_0^1 \underset{\sim}{s}(\mu, \mu') \underset{\sim}{Q}(\mu') \frac{d\mu'}{\mu'} \right] \underset{\sim}{D} \\ & \left[\underset{\sim}{Q}(\mu_0) + \frac{1}{2} \int_0^1 \underset{\sim}{Q}(-\mu') \underset{\sim}{s}(\mu', \mu_0) \frac{d\mu'}{\mu'} \right]. \end{aligned} \quad (38)$$

If we define 2 x 2 matrices $\underset{\sim}{h}(\mu)$ and $\underset{\sim}{\ell}(\mu)$ by

$$\int_{-1}^1 \underset{\sim}{\Psi}(x, \mu) d\mu = \int_0^1 \underset{\sim}{h}(\mu) \underset{\sim}{\Psi}(x, \mu) d\mu \quad (39)$$

and

$$\int_{-1}^1 \underset{\sim}{\Psi}(x, \mu) \mu d\mu = \int_0^1 \underset{\sim}{\ell}(\mu) \underset{\sim}{\Psi}(x, \mu) d\mu, \quad (40)$$

then we can write these matrices in terms of $\underset{\sim}{s}$ -matrix as

$$\underset{\sim}{h}(\mu) = \underset{\sim}{I} + \frac{1}{2} \int_0^1 \underset{\sim}{s}(\mu', \mu) \frac{d\mu'}{\mu'} \quad (41)$$

and

$$\tilde{\ell}(\mu) = \mu \tilde{\Sigma} - \frac{1}{2} \int_0^1 \tilde{s}(\mu', \mu) d\mu' . \quad (42)$$

Equation (38) can now be written, in terms of \tilde{h} and $\tilde{\ell}$ matrices as

$$\frac{1}{\mu} \tilde{\Sigma} \tilde{s}(\mu, \mu_0) + \frac{1}{\mu_0} \tilde{s}(\mu, \mu_0) \tilde{\Sigma} = 2 \tilde{\Phi}(\mu, D \tilde{\Psi}(\mu_0)) , \quad (43)$$

where we have defined 2 x 4 matrices $\tilde{\Psi}(\mu)$ and $\tilde{\Phi}(\mu)$ by

$$\tilde{\Psi}(\mu) = \begin{bmatrix} \tilde{h}(\mu) & \tilde{\ell}(\mu) \end{bmatrix} \quad (44)$$

and

$$\tilde{\Phi}(\mu) = \begin{bmatrix} \tilde{h}_a(\mu) & \tilde{\ell}_a(\mu) \end{bmatrix} . \quad (45)$$

Here the subscript a is used to denote adjoint functions based on the adjoint equation. We have also used the reciprocity relation of the \tilde{s} -matrix,

$$\tilde{S}(\mu, \mu_0) = \tilde{s}_a(\mu_0, \mu) , \quad (46)$$

which can easily be shown in the usual manner [7]. If we now define

$$\tilde{S}(\mu, \mu_0) = \begin{bmatrix} s_{11}(\sigma\mu, \sigma\mu_0) & s_{12}(\sigma\mu, \mu_0) \\ s_{21}(\mu, \sigma\mu_0) & s_{22}(\mu, \mu_0) \end{bmatrix} , \quad (47)$$

$$\tilde{H}(\mu) = \begin{bmatrix} h_{11}(\sigma\mu) & h_{12}(\sigma\mu) \\ h_{21}(\mu) & h_{22}(\mu) \end{bmatrix} \quad (48)$$

and

$$\tilde{L}(\mu) = \begin{bmatrix} \ell_{11}(\sigma\mu) & \ell_{12}(\sigma\mu) \\ \ell_{21}(\mu) & \ell_{22}(\mu) \end{bmatrix}. \quad (49)$$

where s_{ij} , h_{ij} and ℓ_{ij} are the elements of the \tilde{s} , \tilde{h} and $\tilde{\ell}$ matrices, respectively, then we can rewrite Equation (43) as

$$\tilde{s}(\mu, \mu_0) = \frac{2\mu\mu_0}{\mu + \mu_0} \tilde{\Phi}^*(\mu) \tilde{D} \tilde{\Psi}^*(\mu_0), \quad (50)$$

where

$$\tilde{\Psi}^*(\mu) = \begin{bmatrix} H(\mu) & L(\mu) \\ \tilde{\sim} & \tilde{\sim} \end{bmatrix} \quad (51)$$

and

$$\tilde{\Phi}^*(\mu) = \begin{bmatrix} H_a(\mu) & -L_a(\mu) \\ \tilde{\sim} & \tilde{\sim} \end{bmatrix}. \quad (52)$$

From Equations (41), (42), and (50), using Equations (47), (48) and (49), we obtain non-linear integral equations for $\tilde{\Psi}^*(\mu)$ and $\tilde{\Phi}^*(\mu)$:

$$\tilde{\Psi}^*(\mu) = \tilde{Q}^*(\mu) + \mu \tilde{\Psi}^*(\mu) \tilde{D} \int_0^1 \tilde{\Phi}^*(\mu') \tilde{\Theta}(\mu') \tilde{Q}^*(-\mu') \frac{d\mu'}{\mu' + \mu} \quad (53)$$

and

$$\tilde{\Phi}^*(\mu) = \tilde{Q}^*(-\mu) + \mu \tilde{\Phi}^*(\mu) \tilde{D} \int_0^1 \tilde{\Psi}^*(\mu') \tilde{\Theta}(\mu') \tilde{Q}^*(\mu') \frac{d\mu'}{\mu' + \mu}. \quad (54)$$

Equations (53) and (54) are four coupled equations for $\tilde{H}(\mu)$, $\tilde{L}(\mu)$, $\tilde{H}_a(\mu)$ and $\tilde{L}_a(\mu)$. It can easily be shown, however, that these matrices are related by

$$\mu \tilde{H}(\mu) \begin{bmatrix} I & -\tilde{C} \tilde{H}_a \\ \tilde{\sim} & \tilde{\sim} \end{bmatrix} = \tilde{L}(\mu) \begin{bmatrix} \tilde{\Sigma}^{-1} & -\mu \tilde{B} \tilde{L}_a \\ \tilde{\sim} & \tilde{\sim} \end{bmatrix} \quad (55)$$

and

$$\mu \tilde{H}_a(\mu) \left[\tilde{I} - \tilde{C} \tilde{H}_0 \right] = \tilde{L}_a(\mu) \left[\tilde{\Sigma}^{-1} - \mu \tilde{B} \tilde{L}_0 \right], \quad (56)$$

where we have defined moments of a matrix $\tilde{G}(\mu)$ by

$$\tilde{G}_\alpha = \int_0^1 \tilde{\Theta}(\mu) \tilde{G}(\mu) \mu^\alpha d\mu \quad (57)$$

for $\tilde{G}(\mu) = \tilde{H}(\mu), \tilde{L}(\mu), \tilde{H}_a(\mu)$ or $\tilde{L}_a(\mu)$.

The non-linear integral equations for these matrices can, therefore, be reduced to two integral equations,

$$\tilde{H}(\mu) = \tilde{I} + \mu \tilde{H}(\mu) \tilde{C} \int_0^1 \tilde{H}_a(\mu') \tilde{\Theta}(\mu') \frac{d\mu'}{\mu'+\mu} - \mu \tilde{L}(\mu) \tilde{B} \int_0^1 \tilde{L}_a(\mu') \tilde{\Theta}(\mu') \frac{d\mu'}{\mu'+\mu} \quad (58)$$

and

$$\tilde{H}_a(\mu) = \tilde{I} + \mu \tilde{H}_a(\mu) \tilde{C} \int_0^1 \tilde{H}(\mu') \tilde{\Theta}(\mu') \frac{d\mu'}{\mu'+\mu} - \mu \tilde{L}_a(\mu) \tilde{B} \int_0^1 \tilde{L}(\mu') \tilde{\Theta}(\mu') \frac{d\mu'}{\mu'+\mu}. \quad (59)$$

and the constraints, Equations (55) and (56).

Equations (39) and (40), written as

$$\int_0^1 \tilde{h}(\mu) \tilde{F}(\xi, \mu) d\mu = \tilde{M}(\xi) \quad (60)$$

and

$$\int_0^1 \tilde{g}(\mu) \tilde{F}(\xi, \mu) d\mu = \xi \tilde{\chi} \tilde{M}(\xi), \quad (61)$$

for $\xi = \nu$, or $\nu \in (0,1)$, yield, when Equations (12), (25), and (26) are substituted, singular integral equations for $\tilde{H}(\mu)$ and $\tilde{L}(\mu)$, written here collectively as

$$\tilde{\Psi}^*(\nu) \tilde{\lambda}(\nu) = \tilde{\Gamma}(\nu) + \nu \rho \int_0^1 \tilde{\Psi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu-\nu} D\tilde{\Gamma}(\nu), \quad \nu \in (0, \frac{1}{\sigma}), \quad (62)$$

$$\tilde{\Psi}^*(\nu) \tilde{\lambda}(\nu) \tilde{M}^{(2)}(\nu) = \left[\tilde{\Gamma}(\nu) + \nu \rho \int_0^1 \tilde{\Psi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu-\nu} D\tilde{\Gamma}(\nu) \right] \tilde{M}^{(2)}(\nu), \quad \nu \in (\frac{1}{\sigma}, 1), \quad (63)$$

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and

$$\left[\tilde{\Gamma}(\nu_i) + \nu_i \int_0^1 \tilde{\Psi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu - \nu_i} \tilde{D} \tilde{\Gamma}(\nu_i) \right] \tilde{M}(\nu_i) = \tilde{0},$$

$$i = 1, 2, \dots, \kappa. \quad (64)$$

Similar equations can, of course, be obtained for the adjoint matrices. We now extend the definition of the \tilde{H} , \tilde{L} , \tilde{H}_a and \tilde{L}_a matrices to the complex plane by

$$\tilde{\Psi}^*(z) \tilde{\Lambda}(z) = \tilde{\Gamma}(z) + z \int_0^1 \tilde{\Psi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu - z} \tilde{D} \tilde{\Gamma}(z) \quad (65)$$

and

$$\tilde{\Phi}^*(z) \tilde{\Lambda}_a(z) = \tilde{\Gamma}_a(-z) + z \int_0^1 \tilde{\Phi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu - z} \tilde{D} \tilde{\Gamma}_a(z), \quad (66)$$

and thus we can rewrite Equations (53) and (54) as

$$\tilde{\Psi}^*(z) = \tilde{Q}^*(z) + z \tilde{\Psi}^*(z) \tilde{D} \int_0^1 \tilde{\Phi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu + z} \quad (67)$$

and

$$\tilde{\Phi}^*(z) = \tilde{Q}^*(-z) + z \tilde{\Phi}^*(z) \tilde{D} \int_0^1 \tilde{\Psi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\mu + z}. \quad (68)$$

Equation (65), with $z \rightarrow -z$, multiplied from the left by $\tilde{\Phi}^*(z) \tilde{D}$ yields, on using Equation (68), a factorization of $\tilde{\Lambda}(z)$,

$$\tilde{\Phi}^*(z) \tilde{D} \tilde{\Psi}^*(-z) \tilde{\Lambda}(z) = \tilde{Q}^*(z) \tilde{D} \tilde{\Gamma}(z), \quad (69)$$

and, in a similar manner, $\tilde{\Lambda}_a(z)$ can be factored as

$$\tilde{\Psi}^*(z) \tilde{D} \tilde{\Phi}^*(-z) \tilde{\Lambda}_a(z) = \tilde{Q}^*(z) \tilde{D} \tilde{\Gamma}_a(z). \quad (70)$$

We note, from Equations (65) and (66), that

$$\tilde{\Omega}(z) = \tilde{\Psi}^*(-z)\tilde{\Lambda}(z) \quad (71)$$

and

$$\tilde{\Omega}_a(z) = \tilde{\Psi}^*(-z)\tilde{\Lambda}_a(z) \quad (72)$$

are non-singular for $z = \mu \in (0,1)$ and $z = \nu_i$ and thus we conclude from Equations (55), (56), (69) and (70), that $\tilde{H}_a, \tilde{H}_a, \tilde{L}$ and \tilde{L}_a matrices are singular at $z = -\mu, \mu \in (0,1)$, and $z = -\nu_i$.

We would now like to prove our half-range orthogonality theorem:

Theorem: The eigenvectors $\tilde{F}(\nu_i, \mu), \tilde{F}_1^{(1)}(\nu, \mu), \tilde{F}_2^{(1)}(\nu, \mu)$ and $\tilde{F}^{(2)}(\nu, \mu), \nu \in (0,1)$, are orthogonal on the half-range, $\mu \in (0,1)$, to the related set $\tilde{G}(\nu_i, \mu), \tilde{G}_1^{(1)}(\nu, \mu), \tilde{G}_2^{(1)}(\nu, \mu)$ and $\tilde{G}^{(2)}(\nu, \mu), \nu \in (0,1)$, in the sense that

$$\int_0^1 \tilde{G}(\xi, \mu) \tilde{F}(\xi, \mu) \mu d\mu = 0, \quad \xi \neq \xi'; \quad \xi, \xi' = \nu_i \quad \text{or} \quad \nu \in (0,1). \quad (73)$$

Here

$$\tilde{G}(\xi, \mu) = \tilde{E}(\xi, \mu) \tilde{\Psi}(\mu) \tilde{D} \tilde{\Omega}_a(\xi) \tilde{M}_a(\xi), \quad (74)$$

and $\tilde{E}(\xi, \mu)$ is given explicitly by Equations (16), (25) and (26) for $\xi = \nu_i, \xi = \nu \in (0, \frac{1}{\sigma})$ and $\xi = \nu \in (\frac{1}{\sigma}, 1)$, respectively. If we enter Equation (8) into the left-hand side of Equation (73) and use the identity

$$\tilde{E}(\xi, \mu) \tilde{E}(\xi', \mu) \mu = \frac{\xi \xi'}{\xi' - \xi} \left[\tilde{E}(\xi, \mu) - \tilde{E}(\xi', \mu) \right], \quad (75)$$

we find that

$$T = \int_0^1 \tilde{G}(\xi, \mu) \tilde{F}(\xi', \mu) \mu d\mu$$

can be written as

$$T = \frac{\xi \xi'}{\xi' - \xi} \tilde{M}_a(\xi) \tilde{\Omega}_a(\xi) \tilde{D} \left[\xi P \int_0^1 \tilde{\Psi}^*(\mu) \tilde{\Theta}(\mu) \tilde{\Omega}^*(\mu) \frac{d\mu}{\xi - \mu} \right]$$

$$\begin{aligned}
& + \omega(\xi) \tilde{\Psi}^*(\xi) \tilde{\Theta}(\xi) \tilde{Q}^*(\xi) \left[\tilde{D} \tilde{\Gamma}(\xi') \tilde{M}(\xi') \right. \\
& \left. - \frac{\xi \xi'}{\xi' - \xi} \tilde{M}_a(\xi) \tilde{\Omega}_a(\xi) \tilde{D} \int_0^1 \tilde{\Psi}(\mu) \tilde{F}(\xi', \mu) d\mu \right], \tag{76}
\end{aligned}$$

where we let $\omega(\xi) = 0$ for $\xi = \nu_j$.

If we multiply Equation (68) from the left by $\tilde{\Lambda}_a(z)$ and let $z = -\xi$, we find that

$$\xi P \int_0^1 \tilde{\Psi}^*(\mu) \tilde{\Theta}(\mu) \tilde{Q}^*(\mu) \frac{d\mu}{\xi - \mu} = \tilde{\Omega}_a(\xi) - \tilde{\Lambda}_a(\xi) \tilde{Q}^*(\xi). \tag{77}$$

We note that, by definition,

$$\int_0^1 \tilde{\Psi}(\mu) \tilde{F}(\xi', \mu) d\mu = \tilde{\Gamma}(\xi') \tilde{M}(\xi') \tag{78}$$

and thus Equation (76) reduces to

$$\begin{aligned}
T = & - \frac{\xi \xi'}{\xi' - \xi} \tilde{M}_a(\xi) \left[\tilde{\Lambda}_a(\xi) - \omega(\xi) \tilde{\Omega}_a(\xi) \tilde{D} \tilde{\Psi}^*(\xi) \tilde{\Theta}(\xi) \right] \\
& \cdot \tilde{Q}^*(\xi) \tilde{D} \tilde{\Gamma}(\xi') \tilde{M}(\xi'). \tag{79}
\end{aligned}$$

If we use Equations (70), (17) and (22), we conclude from Equation (79) that

$$\int_0^1 \tilde{G}(\xi, \mu) \tilde{F}(\xi', \mu) d\mu = 0, \quad \xi \neq \xi'; \quad \xi, \xi' = \nu_j \quad \text{or} \quad \nu \in (0, 1), \tag{80}$$

which proves the theorem.

3 Normalization Integrals

The half-range orthogonality theorem has been established, and we would therefore like to evaluate the necessary normalization integrals, so that all expansion coefficients appearing in Equation (35) can be expressed concisely in terms of integrals of the expansion function $\tilde{f}(\mu)$. Though the full-range orthogonality relations have been reported by Reith and Siewert [6], we would like to summarize both full-range and half-range cases here in terms of the general form of $\tilde{F}(\xi, \mu)$ given by equations (12), (25) and (26).

For the full-range, we write

$$\int_{-1}^1 \tilde{F}_a(\xi', \mu) \tilde{F}(\xi, \mu) \mu d\mu = 0, \quad \xi \neq \xi'; \quad \xi, \xi' = \pm \nu_i \quad \text{or} \quad \in (-1, 1), \quad (81)$$

whereas for the half-range

$$\int_0^1 \tilde{G}(\xi', \mu) \tilde{F}(\xi, \mu) \mu d\mu = 0, \quad \xi \neq \xi'; \quad \xi, \xi' = \nu_i \quad \text{or} \quad \in (0, 1). \quad (82)$$

Here $\tilde{F}(\xi, \mu)$ is any solution of Equation (7) and $\tilde{G}(\xi, \mu)$ is given by Equation (74). Considering the full-range normalization integrals, we find [6]

$$\int_{-1}^1 \tilde{F}_a(\pm \nu_i, \mu) \tilde{F}(\pm \nu_i, \mu) \mu d\mu = \pm S(\nu_i); \quad i = 1, 2, \dots, \kappa, \quad (83a)$$

$$\int_{-1}^1 \tilde{F}_{a\alpha}^{(1)}(\nu', \mu) \tilde{F}_{\beta}^{(1)}(\nu, \mu) \mu d\mu = S_{\alpha}^{(1)}(\nu) \delta(\nu - \nu') \delta_{\alpha\beta}; \quad \nu, \nu' \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right);$$

$$\alpha, \beta = 1, 2, \quad (83b)$$

and

$$\int_{-1}^1 \tilde{F}_a^{(2)}(\nu', \mu) \tilde{F}^{(2)}(\nu, \mu) \mu d\mu = S^{(2)}(\nu) \delta(\nu - \nu');$$

$$\nu, \nu' \in \left(-1, -\frac{1}{\sigma}\right) \cup \left(\frac{1}{\sigma}, 1\right), \quad (83c)$$

whereas for the half-range we conclude that

$$\int_0^1 \tilde{G}(\nu_i, \mu) \tilde{F}(\nu_i, \mu) \mu d\mu = S(\nu_i); \quad i = 1, 2, \dots, \kappa, \quad (84a)$$

$$\int_0^1 \tilde{G}_{\alpha}^{(1)}(\nu', \mu) \tilde{F}_{\beta}^{(1)}(\nu, \mu) \mu d\mu = S_{\alpha}^{(1)}(\nu) \delta(\nu - \nu') \delta_{\alpha\beta}; \quad \nu, \nu' \in \left(0, \frac{1}{\sigma}\right);$$

$$\alpha, \beta = 1, 2, \quad (84b)$$

and

$$\int_0^1 \tilde{G}^{(2)}(\nu', \mu) \tilde{F}^{(2)}(\nu, \mu) \mu d\mu = S^{(2)}(\nu) \delta(\nu - \nu'); \quad \nu, \nu' \in \left(\frac{1}{\sigma}, 1\right), \quad (84c)$$

where $S(\nu_i)$, $S_\alpha^{(1)}(\nu)$ and $S^{(2)}(\nu)$ are given by Reith and Siewert [6].

Though the representation of the two eigenvectors given by Equation (25) for $\nu \in$ Region ① are convenient for proving completeness and orthogonality theorems, we prefer for actual applications the linear combinations

$$\tilde{\Psi}_\alpha^{(1)}(\nu, \mu) = T_{\alpha 1}(\nu) \tilde{F}_1^{(1)}(\nu, \mu) + T_{\alpha 2}(\nu) \tilde{F}_2^{(1)}(\nu, \mu) \quad (85)$$

which yield the tractable solutions [6]

$$\tilde{\Psi}_\alpha^{(1)}(\nu, \mu) = \begin{bmatrix} \frac{\nu P}{\sigma\nu - \mu} \Delta_{1\alpha}(\nu\mu) + \lambda_{1\alpha}(\nu) \delta(\sigma\nu - \mu) \\ \frac{\nu P}{\nu - \mu} \Delta_{2\alpha}(\nu\mu) + \lambda_{2\alpha}(\nu) \delta(\nu - \mu) \end{bmatrix} \quad (86)$$

where $\Delta_{\alpha\beta}(\nu\mu)$ and $\lambda_{\alpha\beta}(\nu)$ are the matrix elements of $\Delta(\nu\mu) = \mathcal{C} + \nu\mu\mathbf{A}$ and $\lambda(\nu)$.

For the additional eigenvectors, we normalize Equations (26) and (12) to obtain

$$\tilde{\Psi}^{(2)}(\nu, \mu) = \begin{bmatrix} \frac{\nu}{\sigma\nu - \mu} \left[\Lambda_{11}(\nu) \Delta_{12}(\nu\mu) - \Lambda_{12}(\nu) \Delta_{11}(\nu\mu) \right] \\ \frac{\nu P}{\nu - \mu} \left[\Lambda_{11}(\nu) \Delta_{22}(\nu\mu) - \Lambda_{12}(\nu) \Delta_{21}(\nu\mu) \right] + \lambda(\nu) \delta(\nu - \mu) \end{bmatrix} \quad (87)$$

and

$$\tilde{\Psi}(\pm\nu_i, \mu) = \begin{bmatrix} \frac{\nu_i}{\sigma\nu_i \mp \mu} \left[\Lambda_{11}(\nu_i) \Delta_{12}(\pm\nu_i, \mu) - \Lambda_{12}(\nu_i) \Delta_{11}(\pm\nu_i, \mu) \right] \\ \frac{\nu_i}{\nu_i \mp \mu} \left[\Lambda_{11}(\nu_i) \Delta_{22}(\pm\nu_i, \mu) - \Lambda_{12}(\nu_i) \Delta_{21}(\pm\nu_i, \mu) \right] \end{bmatrix} \quad (88)$$

where $\Lambda_{\alpha\beta}(\xi)$ are the matrix elements of $\Lambda(\nu_i)$, $\xi = \pm\nu_i$, and $\lambda(\nu)$, $\xi = \nu$, and $\lambda(\nu) = \det \lambda(\nu)$.

For the explicit forms given above, we list the normalization vectors,

$$\tilde{\mathcal{U}}(\xi) = \int_{-1}^1 \tilde{\Psi}(\xi, \mu) d\mu, \quad (89)$$

as

$$\tilde{\cup}(\pm\nu_i) = \begin{bmatrix} -\Lambda_{12}(\nu_i) \\ \Lambda_{11}(\nu_i) \end{bmatrix}, \quad \tilde{\cup}^{(2)}(\nu) = \begin{bmatrix} -\Lambda_{12}(\nu) \\ \Lambda_{11}(\nu) \end{bmatrix}, \quad (90a,b)$$

$$\tilde{\cup}_1^{(1)}(\nu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\cup}_2^{(1)}(\nu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (90c,d)$$

The solutions given by Equations (86), though more concise than the $\tilde{F}_\alpha^{(1)}(\nu, \mu)$ given by Equation (25), are not orthogonal for $\nu = \nu'$; however, as discussed by Siewert and Zweifel [10], a Schmidt-type procedure can be used to construct suitable adjoint vectors. We should now like to summarize our final results regarding the orthogonality relations:

$$\int_{-1}^1 \tilde{\chi}(\xi', \mu) \tilde{\psi}(\xi, \mu) \mu d\mu = 0, \quad \xi \neq \xi'; \quad \xi, \xi' = \pm\nu_i \quad \text{or} \quad \epsilon \in (-1, 1), \quad (91)$$

$$\int_0^1 \tilde{\Theta}(\xi', \mu) \tilde{\psi}(\xi, \mu) \mu d\mu = 0, \quad \xi \neq \xi'; \quad \xi, \xi' = \nu_i \quad \text{or} \quad \epsilon \in (0, 1), \quad (92)$$

and

$$\int_{-1}^1 \tilde{\chi}(\pm\nu_i, \mu) \tilde{\psi}(\pm\nu_i, \mu) \mu d\mu = \pm N(\nu_i), \quad i = 1, 2, \dots, \kappa, \quad (93a)$$

$$\int_{-1}^1 \tilde{\chi}_\alpha^{(1)}(\nu', \mu) \tilde{\psi}_\beta^{(1)}(\nu, \mu) \mu d\mu = N^{(1)}(\nu) \delta(\nu - \nu') \delta_{\alpha\beta};$$

$$\nu, \nu' \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right); \quad \alpha, \beta = 1, 2, \quad (93b)$$

$$\int_{-1}^1 \tilde{\chi}^{(2)}(\nu', \mu) \tilde{\psi}^{(2)}(\nu, \mu) \mu d\mu = N^{(2)}(\nu) \delta(\nu - \nu');$$

$$\nu, \nu' \in \left(-1, -\frac{1}{\sigma}\right) \cup \left(\frac{1}{\sigma}, 1\right), \quad (93c)$$

$$\int_0^1 \tilde{\Theta}(\nu_i, \mu) \tilde{\psi}(\nu_i, \mu) \mu d\mu = N(\nu_i); \quad i = 1, 2, \dots, \kappa, \quad (94a)$$

$$\int_0^1 \tilde{\Theta}_\alpha^{(1)}(\nu', \mu) \tilde{\psi}_\beta^{(1)}(\nu, \mu) \mu d\mu = N^{(1)}(\nu) \delta(\nu - \nu') \delta_{\alpha\beta}; \quad \nu, \nu' \in \left(0, \frac{1}{\sigma}\right), \quad (94b)$$

$$\int_0^1 \tilde{\Theta}^{(2)}(\nu', \mu) \tilde{\Phi}^{(2)}(\nu, \mu) \mu d\mu = N^{(2)}(\nu) \delta(\nu - \nu'); \quad \nu, \nu' \in \left(\frac{1}{\sigma}, 1\right). \quad (94c)$$

Here

$$\tilde{X}(\pm\nu, \mu) = \tilde{\Phi}_a(\pm\nu, \mu), \quad (95a)$$

$$\tilde{X}_{\tilde{\nu}_1}^{(1)}(\nu, \mu) = N_{2,2}(\nu) \tilde{\Phi}_{a1}^{(1)}(\nu, \mu) - N_{1,2}(\nu) \tilde{\Phi}_{a2}^{(1)}(\nu, \mu), \quad (95b)$$

$$\tilde{X}_{\tilde{\nu}_2}^{(1)}(\nu, \mu) = N_{1,1}(\nu) \tilde{\Phi}_{a2}^{(1)}(\nu, \mu) - N_{2,1}(\nu) \tilde{\Phi}_{a1}^{(1)}(\nu, \mu), \quad (95c)$$

$$\tilde{X}_{\tilde{\nu}}^{(2)}(\nu, \mu) = \tilde{\Phi}_a^{(2)}(\nu, \mu), \quad (95d)$$

and for $\xi = \nu_i$ or $\epsilon \in (0, 1)$

$$\tilde{\Theta}(\xi, \mu) = \left[\xi \tilde{K}(\xi, \mu) \tilde{\Psi}(\mu) \tilde{D} \tilde{\Omega}_a(\xi) + \delta(\xi, \mu) \tilde{\lambda}(\xi) \right] \tilde{V}(\xi) \quad (96)$$

where

$$\tilde{V}(\nu_i) = \tilde{U}_a(\nu_i), \quad \tilde{V}^{(2)}(\nu) = \tilde{U}_a^{(2)}(\nu), \quad (97a,b)$$

$$\tilde{V}_{\tilde{\nu}_1}^{(1)}(\nu) = \begin{bmatrix} N_{2,2}(\nu) \\ -N_{1,2}(\nu) \end{bmatrix} \quad \text{and} \quad \tilde{V}_{\tilde{\nu}_2}^{(1)}(\nu) = \begin{bmatrix} -N_{2,1}(\nu) \\ N_{1,1}(\nu) \end{bmatrix}. \quad (97c,d)$$

Finally, the normalization factors and cross-product integrals, $N_{\alpha\beta}(\nu)$, have been evaluated by Reith and Siewert [6] and are given in Appendix C.

In regard to eventual applications, we would like to evaluate the inner-product integrals, for $\xi, \xi' = \nu_i$ or $\epsilon \in (0, 1)$,

$$\int_0^1 \tilde{\Theta}(\xi', \mu) \tilde{\Phi}(-\xi, \mu) \mu d\mu = \frac{\xi \xi'}{\xi + \xi'} \tilde{V}(\xi') \tilde{\Omega}_a(\xi') \tilde{D} \tilde{\Omega}(\xi) \tilde{U}(\xi). \quad (98)$$

Having proved the half-range orthogonality theorem and evaluated all required

normalization integrals, we may express the solutions to expansions of the form

$$\begin{aligned} \tilde{\Psi}(\mu) &= \sum_{i=1}^K A(\nu_i) \tilde{\Psi}(\nu_i, \mu) + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \tilde{\Psi}_1^{(1)}(\nu, \mu) + A_2^{(1)}(\nu) \tilde{\Psi}_2^{(1)}(\nu, \mu) \right] d\nu \\ &+ \int_{1/\sigma}^1 A^{(2)}(\nu) \tilde{\Psi}^{(2)}(\nu, \mu) d\nu, \quad \mu \in (0, 1), \end{aligned} \quad (99)$$

as

$$A(\nu_i) = \frac{1}{N(\nu_i)} \int_0^1 \tilde{\Theta}(\nu_i, \mu) \tilde{\Psi}(\mu) \mu d\mu, \quad (100a)$$

$$A_\alpha^{(1)}(\nu) = \frac{1}{N^{(1)}(\nu)} \int_0^1 \tilde{\Theta}_\alpha^{(1)}(\nu, \mu) \tilde{\Psi}(\mu) \mu d\mu, \quad (100b)$$

and

$$A^{(2)}(\nu) = \frac{1}{N^{(2)}(\nu)} \int_0^1 \tilde{\Theta}^{(2)}(\nu, \mu) \tilde{\Psi}(\mu) \mu d\mu. \quad (100c)$$

4. Half-Space Applications

Having established the necessary analytical formalism, we would now like to solve typical half-space problems.

For the Milne-problem, we seek a diverging (as $x \rightarrow \infty$) solution to Equation (1) such that

$$\lim_{x \rightarrow \infty} \tilde{\Psi}(x, \mu) e^{-x/\nu_1} < \infty \quad (101a)$$

and

$$\tilde{\Psi}(0, \mu) = 0, \quad \mu \in (0, 1). \quad (101b)$$

Here we use the convention that, when $\kappa \geq 2$, ν_1 denotes the dominant eigenvalue defined as the largest eigenvalue if all are real and the smallest (in absolute value) imaginary eigenvalue if at least one is imaginary. Thus we write the solution as

$$\tilde{\Psi}_M(x, \mu) = \tilde{\Phi}(-\nu_1, \mu) e^{x/\nu_1} + \sum_{i=1}^K A(\nu_i) \tilde{\Psi}(\nu_i, \mu) e^{-x/\nu_i}$$

$$\begin{aligned}
& + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \tilde{\Phi}_1^{(1)}(\nu, \mu) + A_2^{(1)}(\nu) \tilde{\Phi}_2^{(1)}(\nu, \mu) \right] e^{-x/\nu} d\nu \\
& + \int_{1/\sigma}^1 A^{(2)}(\nu) \tilde{\Phi}^{(2)}(\nu, \mu) e^{-x/\nu} d\nu, \quad x \geq 0, \quad \mu \in (-1, 1),
\end{aligned} \tag{102}$$

where we have imposed an arbitrary normalization $A(-\nu_1) = 1$. The solution given by Equation (102) clearly satisfies Equation (1) and the condition at infinity. Constraining Equation (102) to meet the free-surface condition, Equation (101b), yields

$$\begin{aligned}
-\tilde{\Phi}(-\nu_1, \mu) &= \sum_{i=1}^{\kappa} A(\nu_i) \tilde{\Phi}(\nu_i, \mu) + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \tilde{\Phi}_1^{(1)}(\nu, \mu) \right. \\
& \quad \left. + A_2^{(1)}(\nu) \tilde{\Phi}_2^{(1)}(\nu, \mu) \right] d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu) \tilde{\Phi}^{(2)}(\nu, \mu) d\nu, \\
\mu &\in (0, 1).
\end{aligned} \tag{103}$$

If we multiply Equation (103) by $\mu \tilde{\Phi}(\nu_i, \mu)$ and integrate over μ from 0 to 1, we find, by the established half-range orthogonality theorem,

$$A(\nu_i) = -\frac{\nu_i \nu_1}{\nu_i + \nu_1} \frac{1}{N(\nu_i)} \tilde{U}_a(\nu_i) \tilde{\Omega}_a(\nu_i) \tilde{D}\tilde{\Omega}(\nu_1) \tilde{U}(\nu_1). \tag{104}$$

In a similar manner we find

$$\begin{aligned}
A(\nu) &= -\frac{\nu \nu_1}{\nu + \nu_1} \frac{1}{N(\nu)} \tilde{V}(\nu) \tilde{\Omega}_a(\nu) \tilde{D}\tilde{\Omega}(\nu_1) \tilde{U}(\nu_1), \\
\nu &\in (0, 1),
\end{aligned} \tag{105}$$

where $\tilde{V}(\nu)$, $\nu \in (0, 1)$, is defined by Equations (97) and

$$N(\nu) = N^{(1)}(\nu) \theta(\nu) + N^{(2)}(\nu) [1 - \theta(\nu)]. \tag{106}$$

For $x = 0$, the \tilde{S} -matrix can be used to simplify Equation (102) to yield the exit distribution for the Milne-problem. We find

$$\tilde{\Psi}_M(0, -\mu) = \tilde{E}(\nu_1, \mu) \tilde{\Phi}(\mu) \tilde{D}\tilde{\Omega}(\nu_1) \tilde{U}(\nu_1), \quad \mu > 0. \tag{107}$$

For the constant source problem, we wish to construct a bounded solution $\Psi_S(x, \mu)$ to

$$\mu \frac{\partial}{\partial x} \Psi(x, \mu) + \sum_{\nu} \Psi(x, \mu) = C \int_{-1}^1 \Psi(x, \mu') d\mu' + \mu B \int_{-1}^1 \Psi(x, \mu') \mu' d\mu' + S, \quad (108)$$

where S is a given constant vector, subject to

$$\Psi_S(0, \mu) = 0, \quad \mu \in (0, 1), \quad (109)$$

and here we consider, of course, non-conservative media. We thus write

$$\begin{aligned} \Psi_S(x, \mu) = & \sum_{i=1}^K A(\nu_i) \Phi(\nu_i, \mu) e^{-x/\nu_i} + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \Phi_1^{(1)}(\nu, \mu) \right. \\ & \left. + A_2^{(1)}(\nu) \Phi_2^{(1)}(\nu, \mu) \right] e^{-x/\nu} d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu) \Phi^{(2)}(\nu, \mu) e^{-x/\nu} d\nu + \Psi_P(x, \mu) \end{aligned} \quad (110)$$

where

$$\Psi_P(x, \mu) = \left[\sum_{\nu} - 2C \right]^{-1} S \quad (111)$$

is the required particular solution to Equation (108). Entering Equation (110) into Equation (109), we obtain

$$\begin{aligned} -\Psi_P(0, \mu) = & \sum_{i=1}^K A(\nu_i) \Phi(\nu_i, \mu) + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \Phi_1^{(1)}(\nu, \mu) \right. \\ & \left. + A_2^{(1)}(\nu) \Phi_2^{(1)}(\nu, \mu) \right] d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu) \Phi^{(2)}(\nu, \mu) d\nu. \end{aligned} \quad (112)$$

Using Equation (100a) with $\lambda(\mu) = -\Psi_P(0, \mu)$, we find

$$A(\nu_i) = -\frac{\nu_i}{N(\nu_i)} \tilde{U}_a(\nu_i) \tilde{\Omega}_a(\nu_i) D \int_0^1 \tilde{\Psi}^*(\mu) \Phi(\mu) \frac{\mu}{\nu_i - \mu} d\mu \sum_{\nu} \Psi_P(0, \mu), \quad (113)$$

which can be reduced, after some partial fraction analysis and use of Equation (54), to

$$A(\nu_i) = -\frac{1}{N(\nu_i)} \tilde{U}_a(\nu_i) \tilde{\Omega}_a(\nu_i) \left[\begin{array}{c} 0 \\ \int \int \left[\sum_{\nu} - 2C \right]^{-1} S \end{array} \right] \quad (114)$$

In a similar manner we find, for $\nu \in (0,1)$,

$$A(\nu) = -\frac{1}{N(\nu)} \tilde{V}(\nu) \tilde{\Omega}_a(\nu) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \tilde{\Sigma} & -2\tilde{C} \end{bmatrix}^{-1} \tilde{S}, \quad (115)$$

which completes the solution.

For the half-space albedo problem, we seek a bounded (as $x \rightarrow \infty$) solution to Equation (1) such that

$$\tilde{\Psi}(0, \mu) = \delta(\mu - \mu_0) \tilde{F}, \quad \mu, \mu_0 \in (0,1), \quad (116)$$

where \tilde{F} is a given constant vector. The solution $\tilde{\Psi}_a(x, \mu)$ can be written as

$$\begin{aligned} \tilde{\Psi}_a(x, \mu) = & \sum_{i=1}^{\kappa} A(\nu_i) \tilde{\Phi}(\nu_i, \mu) e^{-x/\nu_i} + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \tilde{\Phi}_1^{(1)}(\nu, \mu) \right. \\ & \left. + A_2^{(1)}(\nu) \tilde{\Phi}_2^{(1)}(\nu, \mu) \right] e^{-x/\nu} d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu) \tilde{\Phi}^{(2)}(\nu, \mu) e^{-x/\nu} d\nu, \\ & x \geq 0, \quad \mu \in (-1,1), \end{aligned} \quad (117)$$

where the expansion coefficients must satisfy

$$\begin{aligned} \delta(\mu - \mu_0) \tilde{F} = & \sum_{i=1}^{\kappa} A(\nu_i) \tilde{\Phi}(\nu_i, \mu) + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \tilde{\Phi}_1^{(1)}(\nu, \mu) \right. \\ & \left. + A_2^{(1)}(\nu) \tilde{\Phi}_2^{(1)}(\nu, \mu) \right] d\nu + \int_{1/\sigma}^1 A^{(2)}(\nu) \tilde{\Phi}^{(2)}(\nu, \mu) d\nu, \\ & \mu, \mu_0 \in (0,1). \end{aligned} \quad (118)$$

If we multiply Equation (118) by $\mu \tilde{\Phi}(\xi, \mu)$ and integrate over μ from 0 to 1, we find

$$A(\xi) = \frac{\mu_0}{N(\xi)} \tilde{V}(\xi) \tilde{\Omega}_a(\xi) \tilde{D} \tilde{\Psi}(\mu_0) \tilde{E}(\xi, \mu_0) \tilde{F}, \quad (119)$$

where $N(\xi) = N(\nu_i)$ for $\xi = \nu_i$, $N(\xi) = N^{(1)}(\nu)$ for $\xi = \nu \in (0, \frac{1}{\sigma})$ and $N(\xi) = N^{(2)}(\nu)$ for $\xi = \nu \in (\frac{1}{\sigma}, 1)$. For $x \neq 0$, the $\tilde{\mathcal{S}}$ -matrix can be used to simplify Equation (117) to yield the exit distribution:

$$\tilde{\Psi}_a(0, -\mu) = \frac{\mu_0}{\mu + \mu_0} \sum_{\tilde{\nu}}^{-1} \Phi(\mu) \tilde{D} \tilde{\Psi}(\mu_0) \tilde{\Sigma} F, \quad \mu > 0. \quad (120)$$

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PART II - CRITICAL SLAB PROBLEMS

1. Introduction

Having established half-range orthogonality relations concerning the elementary solutions of the two-group neutron-transport equation for linearly anisotropic scattering, we now wish to show that two-group critical problems for slabs with linearly anisotropic scattering can be reduced to a suitable computational form.

We consider an infinite slab with multiplying properties, centered at $x = 0$ and with the half-thickness x_0 , and thus seek a solution to Equation (I-1) (we refer to equations in Part I by (I)) subject to boundary conditions

$$\tilde{\Psi}(x, \mu) = \tilde{\Psi}(-x, -\mu), \quad \mu \in (-1, 1), \quad (1a)$$

and

$$\tilde{\Psi}(x_0, -\mu) = 0, \quad \mu \in (0, 1) \quad (1b)$$

2. Basic Analysis

The general solution to Equation (I-1) has been reported in Part I, and thus we can write

$$\begin{aligned} \tilde{\Psi}(x, \mu) = & \sum_{i=1}^{\kappa} \left[A(\nu_i) \tilde{\Phi}(\nu_i, \mu) e^{-x/\nu_i} + A(-\nu_i) \tilde{\Phi}(-\nu_i, \mu) e^{x/\nu_i} \right] \\ & + \int_0^{1/\sigma} \left[A_1^{(1)}(\nu) \tilde{\Phi}_1^{(1)}(\nu, \mu) e^{-x/\nu} + A_1^{(1)}(-\nu) \tilde{\Phi}_1^{(1)}(-\nu, \mu) e^{x/\nu} \right] d\nu \\ & + \int_0^{1/\sigma} \left[A_2^{(1)}(\nu) \tilde{\Phi}_2^{(1)}(\nu, \mu) e^{-x/\nu} + A_2^{(1)}(-\nu) \tilde{\Phi}_2^{(1)}(-\nu, \mu) e^{x/\nu} \right] d\nu \\ & + \int_{1/\sigma}^1 \left[A^{(2)}(\nu) \tilde{\Phi}^{(2)}(\nu, \mu) e^{-x/\nu} + A^{(2)}(-\nu) \tilde{\Phi}^{(2)}(-\nu, \mu) e^{x/\nu} \right] d\nu. \quad (2) \end{aligned}$$

where the various A's are the expansion coefficients to be determined by Equations (1). In Equation (2), the eigenvectors $\tilde{\Phi}(\xi, \mu)$, $\xi = \pm\nu_i$ or $\pm\nu \in (0, 1)$, are those reported in Part I. We note that κ in Equation (2) denotes the number of \pm pair of zeros, ν_i , of $\Lambda(z) = \det \Lambda(z)$, where $\Lambda(z)$ is defined by Equation (I-13). Though the largest value of κ has not been determined, we consider here only the cases $\kappa = 1$ and $\kappa = 2$.

The solution given by Equation (2) will inherently satisfy the required symmetry condition, Equation (1a), if we take, in general,

$$A(\xi) = A(-\xi) \quad (3)$$

and thus we can write

$$\begin{aligned} \Psi(x, \mu) = & \sum_{i=1}^{\kappa} B(\nu_i) \left[\underset{\sim}{\Phi}(\nu_i, \mu) e^{-(x+x_0)/\nu_i} + \underset{\sim}{\Phi}(-\nu_i, \mu) e^{(x-x_0)/\nu_i} \right] \\ & + \int_0^{1/\sigma} B_1^{(1)}(\nu) \left[\underset{\sim}{\Phi}_1^{(1)}(\nu, \mu) e^{-(x+x_0)/\nu} + \underset{\sim}{\Phi}_1^{(1)}(-\nu, \mu) e^{(x-x_0)/\nu} \right] d\nu \\ & + \int_0^{1/\sigma} B_2^{(1)}(\nu) \left[\underset{\sim}{\Phi}_2^{(1)}(\nu, \mu) e^{-(x+x_0)/\nu} + \underset{\sim}{\Phi}_2^{(1)}(-\nu, \mu) e^{(x-x_0)/\nu} \right] d\nu \\ & + \int_{1/\sigma}^1 B^{(2)}(\nu) \left[\underset{\sim}{\Phi}^{(2)}(\nu, \mu) e^{-(x+x_0)/\nu} + \underset{\sim}{\Phi}^{(2)}(-\nu, \mu) e^{(x-x_0)/\nu} \right] d\nu, \quad (4) \end{aligned}$$

where we have defined, in general,

$$B(\xi) = A(\xi) e^{x_0/\xi} \quad (5)$$

The expression given by Equation (4) is a solution of the transport equation, Equation (1-1), and clearly satisfies the symmetry condition, Equation (1a). If we now impose the remaining boundary condition, Equation (1b), then we find the expansion coefficients must be determined (to within a constant multiple) from

$$\begin{aligned} -\underset{\sim}{F}(\mu) = & \sum_{i=1}^{\kappa} B(\nu_i) \underset{\sim}{\Phi}(\nu_i, \mu) + \int_0^{1/\sigma} \left[B_1^{(1)}(\nu) \underset{\sim}{\Phi}_1^{(1)}(\nu, \mu) \right. \\ & \left. + B_2^{(1)}(\nu) \underset{\sim}{\Phi}_2^{(1)}(\nu, \mu) \right] d\nu + \int_{1/\sigma}^1 B^{(2)}(\nu) \underset{\sim}{\Phi}^{(2)}(\nu, \mu) d\nu, \quad (6) \\ & \mu \in (0, 1). \end{aligned}$$

where we have introduced

$$\begin{aligned} \tilde{F}(\mu) = & \sum_{i=1}^{\kappa} B(\nu_i) \tilde{\Psi}(-\nu_i, \mu) e^{-2x_0/\nu_i} + \int_0^1 \frac{1}{\sigma} \left[B_1^{(1)}(\nu) \tilde{\Phi}_1^{(1)}(-\nu, \mu) \right. \\ & \left. + B_2^{(1)}(\nu) \tilde{\Phi}_2^{(1)}(-\nu, \mu) \right] e^{-2x_0/\nu} d\nu + \int_{1/\sigma}^1 B^{(2)}(\nu) \tilde{\Phi}^{(2)}(-\nu, \mu) e^{-2x_0/\nu} d\nu. \quad (7) \end{aligned}$$

It is clear that Equation (6) is a system of singular integral equations for the desired expansion coefficients; in fact, it was just this system of singular equations that was solved numerically by Bosler and Metcalf [1] to yield the critical thickness of a multiplying slab. Though modern computing facilities may render feasible iterative solutions of singular integral equations, we prefer a more traditional procedure and thus wish to reduce the given system of singular equations to a system of regular, Fredholm-type, equations, which will subsequently be solved iteratively. The half-range orthogonality relations established in Part I can now be used to reduce Equation (6) to a system of non-singular equations.

If we now take the inner product of Equation (6) with each of $\tilde{\Phi}(\nu, \mu)$, $\tilde{\Phi}_\alpha^{(1)}(\nu, \mu)$, $\alpha = 1$ and 2, and $\tilde{\Phi}^{(2)}(\nu, \mu)$, we can write

$$B(\nu_i) = -\frac{1}{N(\nu_i)} \int_0^1 \tilde{\Phi}(\nu_i, \mu) F(\mu) \mu d\mu, \quad (8a)$$

$$B_\alpha^{(1)}(\nu) = -\frac{1}{N^{(1)}(\nu)} \int_0^1 \tilde{\Phi}_\alpha^{(1)}(\nu, \mu) F(\mu) \mu d\mu, \quad \alpha = 1, 2, \quad (8b)$$

and

$$B^{(2)}(\nu) = -\frac{1}{N^{(2)}(\nu)} \int_0^1 \tilde{\Phi}^{(2)}(\nu, \mu) F(\mu) \mu d\mu, \quad (8c)$$

where the half-range adjoint basis $\tilde{\Phi}(\xi, \mu)$ and the normalization factors $N(\nu_i)$, $N^{(1)}(\nu)$ and $N^{(2)}(\nu)$ are those defined in Part I. The integrals in Equations (8) have been evaluated in Part I and given by Equation (I-98), and therefore with the definition

$$\tilde{B}(\nu) = \begin{bmatrix} B_1^{(1)}(\nu) \\ B_2^{(1)}(\nu) \end{bmatrix} \theta(\nu) + B^{(2)}(\nu) \tilde{U}^{(2)}(\nu) [1 - \theta(\nu)], \quad (9)$$

we can reduce Equation (6) to the system

$$B(\nu_i) \cdot \sum_{j=1}^{\kappa} B(\nu_j) \Delta_{ij} e^{2x_0/\nu_j} = -T_i, \quad i = 1, 2, \dots, \kappa, \quad (10a)$$

and

$$\begin{aligned} \tilde{B}(\nu) = & -\frac{\nu}{N(\nu)} \tilde{W}(\nu) \tilde{\Omega}_a(\nu) D \left[\sum_{j=1}^K \tilde{B}(\nu_j) \tilde{\Omega}(\nu_j) \tilde{U}(\nu_j) e^{-2x_0/\nu_j} \frac{\nu_j}{\nu+\nu_j} \right. \\ & \left. + \int_0^1 \tilde{\Omega}(\nu') \tilde{B}(\nu') e^{-2x_0/\nu'} \frac{\nu'}{\nu+\nu'} d\nu' \right]. \end{aligned} \quad (10b)$$

Here we have defined

$$T_i = \frac{\nu_i}{N(\nu_i)} \tilde{U}_a(\nu_i) \tilde{\Omega}_a(\nu_i) D \int_0^1 \tilde{\Omega}(\nu) \tilde{B}(\nu) e^{-2x_0/\nu} \frac{\nu}{\nu+\nu_i} d\nu, \quad (11)$$

$$\tilde{W}(\nu) = \left[\tilde{U}_1^{(1)}(\nu) \tilde{V}_1^{(1)}(\nu) + \tilde{U}_2^{(1)}(\nu) \tilde{V}_2^{(1)}(\nu) \right] \theta(\nu) + \tilde{U}_1^{(2)}(\nu) \tilde{U}_a^{(2)}(\nu) \left[1 - \theta(\nu) \right] \quad (12)$$

and

$$\Delta_{ij} = -(-1)^{i+j} \frac{\nu_i \nu_j}{\nu_i + \nu_j} \frac{1}{N(\nu_i)} \tilde{U}_a(\nu_i) \tilde{\Omega}_a(\nu_i) D \tilde{\Omega}(\nu_j) \tilde{U}(\nu_j). \quad (13)$$

The vectors $\tilde{U}(\xi)$ and $\tilde{V}(\xi)$, the matrix $\tilde{\Omega}(\xi)$, and their adjoints are those defined in Part I, and we maintain here the convention that, for the case $\kappa = 2$, ν_1 denotes the dominant eigenvalue.

For the case $\kappa = 1$, Equation (10a) reduces, with the arbitrary normalization $B(\nu_1) = \exp(x_0/\nu_1)$, to

$$e^{x_0/\nu_1} \left[1 - \Delta_{11} e^{-2x_0/\nu_1} \right] = -T_1. \quad (14)$$

Equation (10b) is a regular Fredholm-type equation for the required vector $\tilde{B}(\nu)$, $\nu \in (0, 1)$, and thus must now be solved subject to the critical constraint given by Equation (14).

For the case $\kappa = 2$, we have coupled equations for $B(\nu_1)$ and $B(\nu_2)$,

$$B(\nu_1) \left[1 - \Delta_{11} e^{-2x_0/\nu_1} \right] + B(\nu_2) \Delta_{12} e^{-2x_0/\nu_2} = -T_1 \quad (15a)$$

and

$$B(\nu_1) \Delta_{21} e^{-2x_0/\nu_1} + B(\nu_2) \left[1 - \Delta_{22} e^{-2x_0/\nu_2} \right] = -T_2. \quad (15b)$$

and thus, with the normalization $B(\nu_1) = \exp(x_0/\nu_1)$, we must have

$$1 - \left[\Delta_{11} e^{-2x_0/\nu_1} + \Delta_{22} e^{-2x_0/\nu_2} \right] + \left[\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21} \right] e^{-2x_0/\nu_1} e^{-2x_0/\nu_2} \\ + T_1 \left[1 - \Delta_{22} e^{-2x_0/\nu_2} \right] e^{-x_0/\nu_1} - T_2 \Delta_{12} e^{-x_0/\nu_1} e^{-2x_0/\nu_2} = 0. \quad (16)$$

Equations (10b) and (15a) or (15b) are coupled regular Fredholm-type equations for the required vector $\hat{B}(\nu)$, $\nu \in (0,1)$, and $B(\nu_2)$ and thus must now be solved subject to the critical constraint given by Equation (16).

3 Computational Results

To initiate our computation, we use the data sets of Bosler and Metcalf¹, given in Table 1 to define $\hat{\Sigma}$, \hat{C} and \hat{B} , which subsequently we consider the basic parameters, in the following manner;

$$c_{ij} = \frac{1}{2\sigma_2} \left[\sigma_{ij}^{(0)} + \chi_i \bar{\nu}_j \sigma_{jt} \right] \quad (17)$$

$$b_{i1} = \frac{3}{2\sigma_2} \sigma_{ij}^{(1)} \quad (18)$$

and

$$\sigma = \sigma_1/\sigma_2 \quad (19)$$

The optical variable x is defined as

$$x = \sigma_2 z \quad (20)$$

where z is the space variable.

¹ Bosler, G.E. and D. R. Metcalf. 1972. Private communication

Table 1
Two-group data sets for critical slabs with anisotropic scattering

Case	Group (i)	σ_i	$\bar{\nu}_i \sigma_{if}$	χ_i	$\sigma_{i's}^{(0)}$	$\sigma_{i's}^{(1)}$	$\sigma_{j'is}^{(0)}$	$\sigma_{j'is}^{(1)}$
I (H ₂ O)	1	2.52025	0.12658	0.0	2.44383	0.83318	0.0	0.0
	2	0.65696	0.002621	1.0	0.62568	0.27459	0.029227	0.0075737
II (D ₂ O)	1	0.54628	0.2425	0.0	0.42410	0.05439	0.0	0.0
	2	0.33588	0.007043	1.0	0.31980	0.06694	0.004555	-0.0003972

Of course, before we can proceed to solve Equation (10b) with Equation (14) or with Equations (15) and (16), we must compute the discrete eigenvalues ν_i and the required matrices H_i , K_i and their adjoints. Calculations of ν_i and the matrices are given in detail in Appendices A and B, respectively. In Table 2 we list the discrete eigenvalues for the considered cases.

Table 2

Discrete eigenvalues

Case	ν_1	ν_2
I	i4.624021	1.042005
II	i199.2944	

We note that here all the matrices are complex and thus all of our calculations were performed in complex mode (and in double precision) on an IBM 370/165 machine using the improved Gaussian quadrature scheme. Several numerical checks were carried out once the matrices were established and thus we believe our results given in Tables 3 and 4 for case I are accurate to within the usual round-off convention.

Equations (14) and (15) are, of course, amenable to approximations. If we set $\mathcal{B}(\nu) = \mathcal{Q}$, for the case $\kappa = 1$, and $\mathcal{B}(\nu) = \mathcal{Q}$ and $B(\nu_2) = 0$, for the case $\kappa = 2$, we find the first order approximation (x_1) of the critical dimension to be given by

$$1 - \Delta_{11} e^{-2x_1/\nu_1} = 0 \quad (21)$$

For the case $\kappa = 2$, if we set only $\mathcal{B}(\nu) = \mathcal{Q}$, we obtain the second order approximation (x_2) given by

$$1 - \Delta_{11} e^{-2x_2/\nu_1} - \Delta_{22} e^{-2x_2/\nu_2} + \left[\Delta_{11} \Delta_{22} - \Delta_{12} \Delta_{21} \right] e^{-2x_2/\nu_1} e^{-2x_2/\nu_2} = 0. \quad (22)$$

Table 3

\tilde{H} and \tilde{H}_a matrices for Case I (H_2O)

μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	1.0	0.0	0.0	1.0
0.1	1.60309 + i0.01692	0.31094 + i0.11529	0.01904 + i0.00484	1.21314 + i0.03299
0.2	2.00533 + i0.04545	0.75150 + i0.30977	0.03851 + i0.01133	1.37781 + i0.07723
0.3	2.31840 + i0.08425	1.28033 + i0.57418	0.05859 + i0.01933	1.52664 + i0.13174
0.4	2.56674 + i0.13247	1.86986 + i0.90278	0.07891 + i0.02876	1.66477 + i0.19601
0.5	2.76404 + i0.18942	2.49994 + i1.29090	0.09914 + i0.03956	1.79400 + i0.26962
0.6	2.91945 + i0.25449	3.15516 + i1.73439	0.11922 + i0.05167	1.91506 + i0.35213
0.7	3.03957 + i0.32712	3.82341 + i2.22938	0.13882 + i0.06502	2.02822 + i0.44309
0.8	3.12930 + i0.40677	4.49496 + i2.77221	0.15784 + i0.07953	2.13349 + i0.54204
0.9	3.19244 + i0.49291	5.16188 + i3.35930	0.17615 + i0.09515	2.23080 + i0.64845
1.0	3.23196 + i0.58504	5.81766 + i3.98713	0.19367 + i0.11178	2.32002 + i0.76182
μ	$H_{a11}(\mu)$	$H_{a12}(\mu)$	$H_{a21}(\mu)$	$H_{a22}(\mu)$
0.0	1.0	0.0	0.0	1.0
0.1	1.60309 + i0.01691	0.07147 + i0.02733	0.08269 + i0.02042	1.21313 + i0.03301
0.2	2.00532 + i0.04549	0.17260 + i0.07352	0.16736 + i0.04780	1.37779 + i0.07727
0.3	2.31832 + i0.08441	0.29381 + i0.13644	0.25469 + i0.08153	1.52662 + i0.13177
0.4	2.56652 + i0.13286	0.42872 + i0.21475	0.34318 + i0.12128	1.66475 + i0.19604
0.5	2.76359 + i0.19016	0.57263 + i0.30737	0.43161 + i0.16681	1.79399 + i0.26962
0.6	2.91868 + i0.25572	0.72196 + i0.41333	0.51904 + i0.21784	1.91507 + i0.35210
0.7	3.03836 + i0.32895	0.87388 + i0.53170	0.60470 + i0.27409	2.02825 + i0.44302
0.8	3.12753 + i0.40933	1.02612 + i0.66163	0.68796 + i0.33526	2.13356 + i0.54191
0.9	3.18997 + i0.49633	1.17682 + i0.80225	0.76831 + i0.40106	2.23092 + i0.64825
1.0	3.22864 + i0.58943	1.32446 + i0.95272	0.84536 + i0.47115	2.32020 + i0.76154

Table 4

Moments of the matrices for Case I

G	G_{011}	G_{012}	G_{021}	G_{022}
H	$0.44289 + i0.00719$	$0.12141 + i0.04897$	$0.09834 + i0.04500$	$1.75612 + i0.30667$
H_a	$0.44289 + i0.00719$	$0.02789 + i0.01162$	$0.42833 + i0.18970$	$1.75614 + i0.30663$
L	$0.04388 - i0.00410$	$-0.06214 - i0.02823$	$-0.05471 - i0.02570$	$0.13658 - i0.17681$
L_a	$0.04388 - i0.00410$	$-0.01426 - i0.00670$	$-0.23837 - i0.10831$	$0.13658 - i0.17681$

Though we have not been able to determine in general whether or not Equations (21) and (22) yield real x_1 and x_2 , we report in Table 5 numerical solutions of Equations (21) and (22), for the considered cases, along with the critical dimension (x_{01}) obtained by P-1 approximation and our exact value obtained by iteration. Finally, in Table 6, we list the discrete expansion coefficients and the critical dimensions in cm along with those reported by Bosler and Metcalf[1], and in Table 7 the flux distributions in the slab are given. The basic test of the results is how accurately the boundary condition, Equation (1b), in addition to the critical condition, is satisfied. Since the distributional nature of the continuum eigenvectors limits the accuracy of a pointwise verification of the boundary condition, we have chosen to consider, instead, various moments of Equation (6). To illustrate these checks, we consider Equation (6) written symbolically as $\mathcal{L}(\mu) = R(\mu)$, $\mu \in (0,1)$. We have computed $\int_0^1 \mathcal{L}(\mu) \mu^k d\mu$ and $\int_0^1 R(\mu) \mu^k d\mu$, $k = 1, 2, \dots, 30$, and found agreement consistent with the data reported herein. Although the above checks do not rigorously guarantee the accuracy of our results, we believe that the number and diversity of these checks provide a reasonable degree of confidence in our reported values.

Table 5

Exact and approximate critical slab dimensions

Case	x_{01}	x_1	x_2	x_0
I	6.3580	6.2384 + i0.0	6.2384 + i0.0	6.2384
II	312.24	312.18 + i0.0		312.18

Table 6

Critical slab results

Case	$A(\nu_1)$	$A(\nu_2)$	z_0 (cm)	z_0^* (cm)
I	1.0	-0.000626	9.4959	9.4915
II	1.0		929.45	1000.5

* Bosler and Metcalf [1]

Table 7
Flux distributions

x/x_0	Case I		Case II	
	$\psi_1(x)$	$\psi_2(x)/\psi_1(0)$	$\psi_1(x)$	$\psi_2(x)/\psi_1(0)$
0.0	0.015574	2.7601	0.014992	26.824
0.1	0.015431	2.7351	0.014808	26.495
0.2	0.015002	2.6603	0.014262	25.518
0.3	0.014294	2.5373	0.013367	23.916
0.4	0.013314	2.3682	0.012144	21.728
0.5	0.012071	2.1562	0.010624	19.008
0.6	0.010569	1.9052	0.008844	15.823
0.7	0.008806	1.6195	0.006847	12.250
0.8	0.006757	1.3038	0.004682	8.378
0.9	0.004357	0.9589	0.002403	4.300
1.0	0.001195	0.5227	0.000036	0.096

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APPENDIX A

Closed-form Solution of $\Lambda(z) = 0$

A class of transcendental equations have recently been solved in closed form using the method developed by Burniston and Siewert [1]. The method consists in first converting the equation into an equivalent equation, $\Omega(z) = 0$, where $\Omega(z)$ is a multi-valued function of a complex variable. From this function a sequence of branches, which may be finite or infinite in number, is chosen such that the zeros of each branch correspond to solutions of the original equations. The argument principle is then used to determine how many zeros each branch has for the relevant ranges of the parameters involved. The next step is to establish factorization equations which express each branch in terms of the zeros of the branch and a non-vanishing sectionally analytic function. This function is derived from Riemann boundary-value problems obtained from the factorization equations. The factorization equations so expressed are evaluated at appropriate points resulting in a set of polynomial equations for the roots, which are then solved in the usual manner.

We report here the exact closed-form solution of $\Lambda(z) = 0 : \Lambda(z) = \det \hat{\Lambda}(z)$ where $\hat{\Lambda}(z)$ is given by equation (1-13).

Following the argument similar to Case and Zweifel (2) it can be shown that

$$\Omega(z) \triangleq \left[X(z)X(-z) \right]^{-1} \Lambda(z)\Lambda^{-1}(\infty) = \prod_{i=1}^{\kappa} (\nu_i^2 - z^2) \quad (\text{A.1})$$

where

$$X(z) = \frac{1}{(1-z)^\kappa} \exp \left[\frac{1}{\pi} \int_0^1 \arg \Lambda^+(\mu) \frac{d\mu}{\mu-z} \right]. \quad (\text{A.2})$$

To find the discrete eigenvalues ν_i from Equation (A.1), we need to determine the number of pairs of roots κ . Siewert and Shieh [5] have determined the number and kind of the discrete eigenvalues for the case of $\mathcal{B} = \mathcal{Q}$ in terms of the basic parameters \mathcal{X} and \mathcal{C} . Although, for the case considered here the number and kind of the eigenvalues or the maximum κ has not been determined in general, κ can easily be determined, numerically, for a given set of parameters, by the argument principle as

$$\kappa = \frac{1}{\pi} \Delta_{0,1} \arg \Lambda^+(\mu). \quad (\text{A.3})$$

where $\Delta_{0,1} \arg \Lambda^+(\mu)$ denotes the change of argument of $\Lambda^+(\mu)$ when μ changes from 0 to 1. Once κ is found, Equation (A.1) is evaluated at appropriate points resulting in a set of equations for ν_i , which can be solved in the usual manner. Thus, for the case $\kappa = 1$, we find, taking $z = i$ in Equation (A.1),

$$\nu_1^2 = -1 + 2 \exp\left[-\frac{2}{\pi} \int_0^1 \arg \Lambda^+(\mu) \frac{\mu}{\mu^2+1} d\mu\right] \Lambda(i) \Lambda^{-1}(\infty),$$

$$\kappa = 1, \quad (\text{A.4})$$

and for $\kappa = 2$, we evaluated Equation (A.1) at two points, $z = i$ and $z = 2i$, and thus we obtain

$$\nu_1^2 = \frac{1}{2} \left[R - (R^2 - 4S)^{1/2} \right] \quad (\text{A.5a})$$

and

$$\nu_2^2 = \frac{1}{2} \left[R + (R^2 - 4S)^{1/2} \right] \quad (\text{A.5b})$$

where

$$R = -5 - \frac{1}{3} \left[\Omega(i) - \Omega(2i) \right] \quad (\text{A.6a})$$

and

$$S = 4 + \frac{1}{3} \left[4\Omega(i) - \Omega(2i) \right]. \quad (\text{A.6b})$$

We have evaluated the integrals in Equations (A.4) or (A.6) using the improved Gaussian quadrature scheme to obtain ν_i , and the result was further refined by using Newton's iteration scheme.

APPENDIX B

Computation of \tilde{H} , \tilde{L} , and Associated Adjoint Matrices

As discussed in the main text, proper evaluation of the \tilde{H} , \tilde{L} , and the associated adjoint matrices is of primary importance to any half-space problem defined by the considered two-group transport equations. Here we would like, therefore, to report the procedure we used to compute these matrices.

Though the maximum number of the discrete eigenvalues has not been determined, we consider only the cases $\kappa = 1$ and $\kappa = 2$.

Thus, we consider solutions of the coupled integral equations, Equations (I-58) and (I-59), with constraints, Equations (I-55), (I-56) and (I-64).

We first rewrite Equation (I-59) using Equation (I-56) as

$$\begin{aligned} \tilde{H}_a^{-1}(\mu) = & \int_{\tilde{\nu}} \mu \tilde{C} \int_0^1 \tilde{H}(\mu') \tilde{\Theta}(\mu') \frac{d\mu'}{\mu' + \mu} \\ & + \mu^2 \left[\int_{\tilde{\nu}} \tilde{C} \tilde{H}_0 \right] \left[\int_{\tilde{\nu}} \tilde{\Sigma}^{-1} \mu \tilde{B} \tilde{L}_0 \right]^{-1} \tilde{B} \int_0^1 \tilde{L}(\mu') \tilde{\Theta}(\mu') \frac{d\mu'}{\mu' + \mu} \end{aligned} \quad (\text{A } 7)$$

If we rewrite the discrete constraint of \tilde{H} , Equation (I-64), as

$$\nu \int_0^1 \tilde{H}(\mu') \tilde{\Theta}(\mu') \frac{d\mu'}{\nu_1 + \mu} \left[\tilde{C} + \nu_1^2 \tilde{\Sigma} \tilde{A} \right] \tilde{U}(\nu) = \left[\tilde{I} + \nu_1^2 \tilde{H}_0 \tilde{\Sigma} \tilde{A} \right] \tilde{U}(\nu) \quad (\text{A } 8)$$

and define, for the case $\kappa = 1$,

$$\tilde{\Delta}(\mu) = \begin{bmatrix} \frac{\nu_1(1+\mu)}{\nu_1 + \mu} & 0 \\ 0 & 1 \end{bmatrix} \quad \kappa = 1, \quad (\text{A } 9)$$

and

$$\tilde{V} = \left[\left[\tilde{C} + \nu_1^2 \tilde{\Sigma} \tilde{A} \right] \tilde{U}(\nu_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right], \quad \kappa = 1, \quad (\text{A } 10)$$

then the first integral term in Equation (A.7), multiplied by $\tilde{V} \tilde{\Delta}(\mu)$ from the right, can be

reduced to

$$\int_0^1 \frac{\tilde{H}(\mu') \tilde{\Theta}(\mu')}{\tilde{\mu}' + \mu} \frac{d\mu'}{\tilde{\mu}' + \mu} \tilde{V} \tilde{\Delta}(\mu) = \int_0^1 \frac{\tilde{H}(\mu') \tilde{\Theta}(\mu') \tilde{K}(\mu')}{\tilde{\mu}' + \mu} \frac{d\mu'}{\tilde{\mu}' + \mu} - \frac{1-\nu_1}{\nu_1 + \mu} \left[\tilde{I} + \nu_1^2 \tilde{H}_0 \tilde{\Sigma} \tilde{A} \right] \begin{bmatrix} \tilde{U}(\nu_1) & \tilde{O} \\ \tilde{\nu} & \tilde{\nu} \end{bmatrix}. \quad (\text{A.11})$$

where $\tilde{K}(\mu) = \tilde{V} \tilde{\Delta}(-\mu)$, after invoking the constraint, Equation (A.8), and using the identity

$$\tilde{\Delta}^{-1}(-\mu') \tilde{\Delta}(\mu) = \tilde{I} - \frac{\nu_1(1-\nu_1)(\mu+\mu')}{(\nu_1-\mu')(\nu_1+\mu)} \tilde{\Delta}^{-1}(-\mu') \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (\text{A.12})$$

The second integral term in Equation (A.7) can also be reduced to a similar form using the constraint of \tilde{L} . If we define, for the case $\kappa = 2$,

$$\tilde{\Delta}(\mu) = \begin{bmatrix} \frac{\nu_1(1+\mu)}{\nu_1+\mu} & 0 \\ 0 & \frac{\nu_2(1+\mu)}{\nu_2+\mu} \end{bmatrix}, \quad \kappa = 2, \quad (\text{A.13})$$

and

$$\tilde{V} = \begin{bmatrix} \left[\tilde{C} + \nu_1^2 \tilde{\Sigma} \tilde{A} \right] \tilde{U}(\nu_1) & \left[\tilde{C} + \nu_2^2 \tilde{\Sigma} \tilde{A} \right] \tilde{U}(\nu_2) \\ \tilde{\nu} & \tilde{\nu} \end{bmatrix}, \quad \kappa = 2, \quad (\text{A.14})$$

then Equation (A.7) can be reduced, for both cases $\kappa = 1$ and $\kappa = 2$, to

$$\begin{aligned} \tilde{H}_a^{-1}(\mu) &= \tilde{I} - \mu \tilde{C} \left[\int_0^1 \frac{\tilde{H}(\mu') \tilde{\Theta}(\mu') \tilde{K}(\mu')}{\tilde{\mu}' + \mu} \frac{d\mu'}{\tilde{\mu}' + \mu} + \tilde{P}(\mu) \right] \tilde{R}(\mu) \\ &+ \mu^2 \tilde{S}(\mu) \left[\int_0^1 \frac{\tilde{L}(\mu') \tilde{\Theta}(\mu') \tilde{K}(\mu')}{\tilde{\mu}' + \mu} \frac{d\mu'}{\tilde{\mu}' + \mu} + \tilde{Q}(\mu) \right] \tilde{R}(\mu) \end{aligned} \quad (\text{A.15})$$

Equation (I-58) can be reduced, in the same manner, to

$$\begin{aligned} \tilde{H}_a^{-1}(\mu) &= \tilde{I} - \mu \tilde{C} \left[\int_0^1 \frac{\tilde{H}_a(\mu') \tilde{\Theta}(\mu') \tilde{K}_a(\mu')}{\tilde{\mu}' + \mu} \frac{d\mu'}{\tilde{\mu}' + \mu} + \tilde{P}_a(\mu) \right] \tilde{R}_a(\mu) \\ &+ \mu^2 \tilde{S}_a(\mu) \left[\int_0^1 \frac{\tilde{L}_a(\mu') \tilde{\Theta}(\mu') \tilde{K}_a(\mu')}{\tilde{\mu}' + \mu} \frac{d\mu'}{\tilde{\mu}' + \mu} + \tilde{Q}_a(\mu) \right] \tilde{R}_a(\mu) \end{aligned} \quad (\text{A.16})$$

Here we have defined

$$\underset{\sim}{K}(\mu) = \underset{\sim}{V} \underset{\sim}{\Delta}(-\mu), \quad (\text{A } 17)$$

$$\begin{aligned} \underset{\sim}{P}(\mu) &= \frac{\nu_1 - 1}{\nu_1 + \mu} \left[\underset{\sim}{I} + \nu_1^2 \underset{\sim}{H}_0 \underset{\sim}{\Sigma} \underset{\sim}{A} \right] \begin{bmatrix} \underset{\sim}{U}(\nu_1) & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{U}(\nu_2) \end{bmatrix} \\ &+ (\kappa - 1) \frac{\nu_2 - 1}{\nu_2 + \mu} \left[\underset{\sim}{I} + \nu_2^2 \underset{\sim}{H}_0 \underset{\sim}{\Sigma} \underset{\sim}{A} \right] \begin{bmatrix} \underset{\sim}{O} & \underset{\sim}{U}(\nu_2) \\ \underset{\sim}{O} & \underset{\sim}{U}(\nu_1) \end{bmatrix}, \end{aligned} \quad (\text{A } 18)$$

$$\begin{aligned} \underset{\sim}{Q}(\mu) &= \frac{\nu_1 - 1}{\nu_1 + \mu} \left[\nu_1 \underset{\sim}{\gamma} + \nu_1^2 \underset{\sim}{L}_0 \underset{\sim}{\Sigma} \underset{\sim}{A} \right] \begin{bmatrix} \underset{\sim}{U}(\nu_1) & \underset{\sim}{O} \\ \underset{\sim}{O} & \underset{\sim}{U}(\nu_2) \end{bmatrix} \\ &+ (\kappa - 1) \frac{\nu_2 - 1}{\nu_2 + \mu} \left[\nu_2 \underset{\sim}{\gamma} + \nu_2^2 \underset{\sim}{L}_0 \underset{\sim}{\Sigma} \underset{\sim}{A} \right] \begin{bmatrix} \underset{\sim}{O} & \underset{\sim}{U}(\nu_2) \\ \underset{\sim}{O} & \underset{\sim}{U}(\nu_1) \end{bmatrix}, \end{aligned} \quad (\text{A } 19)$$

$$\underset{\sim}{R}(\mu) = \underset{\sim}{\Delta}^{-1}(\mu) \underset{\sim}{V}^{-1}, \quad (\text{A } 20)$$

and

$$\underset{\sim}{S}(\mu) = \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{C} \underset{\sim}{H}_0 \\ \underset{\sim}{O} & \underset{\sim}{O} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\Sigma}^{-1} & \mu \underset{\sim}{B} \underset{\sim}{L}_0 \\ \underset{\sim}{O} & \underset{\sim}{O} \end{bmatrix}^{-1} \underset{\sim}{B}, \quad (\text{A } 21)$$

and all other quantities are defined in Part I. Solutions of Equations (A.15), (A.16), (I.55) and (I.56) will inherently satisfy the discrete constraint, Equation (I.64). We have solved these equations iteratively using the improved Gaussian quadrature scheme of Kronrod [3]. It is noted that, in one case, the convergence was faster by more than fifteen times, in terms of the number of iterations, with the above procedure, as compared with solving the original non-linear equations.

To check the accuracy of the converged solutions, Equation (I-64) was evaluated to ensure that the solutions did indeed satisfy the constraints. From Equations (I-53) and (I-54) we find a moment relation of the matrices,

$$\begin{bmatrix} \underset{\sim}{\Psi}_0 & \underset{\sim}{I}_4 \\ \underset{\sim}{O} & \underset{\sim}{O} \end{bmatrix} \underset{\sim}{D} \begin{bmatrix} \underset{\sim}{\Phi}_0 & \underset{\sim}{I}_4 \\ \underset{\sim}{O} & \underset{\sim}{O} \end{bmatrix} = \underset{\sim}{D} \begin{bmatrix} \underset{\sim}{I}_4 & 2 \underset{\sim}{Q}_0 \underset{\sim}{D} \\ \underset{\sim}{O} & \underset{\sim}{O} \end{bmatrix}, \quad (\text{A } 22)$$

where

$$\underset{\sim}{\Psi}_0 = \underset{\sim}{D} \int_0^1 \underset{\sim}{Q}^*(\mu) \underset{\sim}{\Theta}(\mu) \underset{\sim}{\Psi}^*(\mu) d\mu, \quad (\text{A } 23a)$$

$$\underset{\sim}{\Phi}_0 = \underset{\sim}{D} \int_0^1 \underset{\sim}{Q}^*(-\mu) \underset{\sim}{\Theta}(\mu) \underset{\sim}{\Phi}^*(\mu) d\mu, \quad (\text{A } 23b)$$

and

$$\hat{D}_0 = \begin{bmatrix} \hat{\Sigma}^{-1} & \hat{0} \\ \hat{0} & \frac{1}{3} \hat{\Sigma}^{-1} \end{bmatrix}, \quad (\text{A.24})$$

and \hat{I}_4 is the 4 x 4 unit matrix. We evaluated Equation (A.22) to find agreement between two sides to more than ten significant figures. The matrices $\hat{H}(z)$, $\hat{L}(z)$, $\hat{L}_a(z)$ and $\hat{L}_b(z)$ have singularities at $z = -\nu_i$, $i = 1, 2, \dots, \kappa$. We evaluated the inverse of each matrix at $z = -\nu_i$, $i = 1, 2, \dots, \kappa$, and found that the determinant was smaller by at least ten orders of magnitude than the smallest element in absolute value. Further, the number of quadrature points was increased successively to ensure that the results did not change with higher quadratures.

APPENDIX C

Dispersion Function and Normalization Integrals

We list here, in explicit form, several functions and various integrals reported by Reith and Siewert [4]. As defined in Part I, we denote the elements of \tilde{C} and \tilde{B} by c_{ij} and b_{ij} , respectively. Similarly the elements of \tilde{A} are a_{ij} and C , B and A are the determinants of \tilde{C} , \tilde{B} and \tilde{A} , respectively. In addition we use the abbreviation.

$$T(z) = \tanh^{-1}(z). \quad (\text{A.25})$$

We write

$$\tilde{\Lambda}(z) = \begin{bmatrix} \Lambda_{11}(z) & \Lambda_{12}(z) \\ \Lambda_{21}(z) & \Lambda_{22}(z) \end{bmatrix}, \quad z \notin (-1, 1) \quad (\text{A.26})$$

$$\tilde{\lambda}(\nu) = \begin{bmatrix} \Lambda_{11}(\nu) & \Lambda_{12}(\nu) \\ \lambda_{21}(\nu) & \lambda_{22}(\nu) \end{bmatrix}, \quad \nu \in (-1, -\frac{1}{\sigma}) \cup (\frac{1}{\sigma}, 1), \quad (\text{A.27})$$

and

$$\tilde{\lambda}(\nu) = \begin{bmatrix} \lambda_{11}(\nu) & \lambda_{12}(\nu) \\ \lambda_{21}(\nu) & \lambda_{22}(\nu) \end{bmatrix}, \quad \nu \in (-\frac{1}{\sigma}, \frac{1}{\sigma}), \quad (\text{A.28})$$

where

$$\Lambda_{1\alpha}(z) = \delta_{1\alpha} + 2a_{1\alpha}z^2 - 2z\Delta_{1\alpha}(\sigma z^2)T(\frac{1}{\sigma z}), \quad (\text{A.29a})$$

$$\Lambda_{2\alpha}(z) = \delta_{2\alpha} + 2a_{2\alpha}z^2 - 2z\Delta_{2\alpha}(z^2)\Pi(\frac{1}{z}), \quad (\text{A.29b})$$

$$\lambda_{1\alpha}(\nu) = \delta_{1\alpha} + 2a_{1\alpha}\nu^2 - 2\nu\Delta_{1\alpha}(\sigma\nu^2)T(\sigma\nu), \quad (\text{A.29c})$$

and

$$\lambda_{2\alpha}(\nu) = \delta_{2\alpha} + 2a_{2\alpha}\nu^2 - 2\nu\Delta_{2\alpha}(\nu^2)T(\nu). \quad (\text{A.29d})$$

The dispersion function $\Lambda(z) = \det \underset{\sim}{\Lambda}(z)$ can be written as

$$\Lambda(z) = 1 + 2z^2P_1(z) - 2zP_2(z)T\left(\frac{1}{\sigma z}\right) - 2zP_3(z)T\left(\frac{1}{z}\right) + 4z^2P_4(z)T\left(\frac{1}{\sigma z}\right)T\left(\frac{1}{z}\right), \quad (\text{A.30})$$

where

$$P_1(z) = a_{11} + a_{22} + 2Az^2, \quad (\text{A.31a})$$

$$P_2(z) = c_{11} + (\sigma a_{11} + 2c_{11}a_{22} - 2c_{12}a_{21})z^2 + 2\sigma Az^4, \quad (\text{A.31b})$$

$$P_3(z) = c_{22} + (a_{22} + 2c_{22}a_{11} - 2c_{21}a_{12})z^2 + 2Az^4, \quad (\text{A.31c})$$

and

$$P_4(z) = C + (\sigma c_{22}a_{11} + c_{11}a_{22} - c_{12}a_{21} - \sigma c_{21}a_{12})z^2 + \sigma Az^4. \quad (\text{A.31d})$$

The boundary values of $\Lambda(z)$ can be expressed as

$$\begin{aligned} \Lambda^\pm(\mu) = & 1 + 2\mu^2P_1(\mu) - 2\mu P_2(\mu)T'(\sigma\mu) - 2\mu P_3(\mu)T(\mu) \\ & + 4\mu^2P_4(\mu)T'(\sigma\mu)T(\mu) - \pi^2\mu^2P_4(\mu)\theta(\mu) \\ & \pm i\pi \left[P_2(\mu)\theta(\mu) + P_3(\mu) - 2\mu P_4(\mu)T(\mu)\theta(\mu) \right. \\ & \left. - 2\mu P_4(\mu)T'(\sigma\mu) \right], \quad \mu \in (-1, 1), \end{aligned} \quad (\text{A.32})$$

where $T'(\sigma\mu) = T(\sigma\mu)$, $\mu \in (-\frac{1}{\sigma}, \frac{1}{\sigma})$; and $T'(\sigma\mu) = T(\frac{1}{\sigma\mu})$, $\mu \in (-1, -\frac{1}{\sigma}) \cup (\frac{1}{\sigma}, 1)$.

The normalization integrals can be summarized as

$$N(\nu_i) = \nu_i^2 \left[P_3(\nu_i) - 2\nu_i P_4(\nu_i)T\left(\frac{1}{\sigma\nu_i}\right) \right] \frac{d}{dz} \Lambda(z) \Big|_{z = \nu_i}, \quad (\text{A.33})$$

$$N^{(1)}(\nu) = \nu \Lambda^+(\nu) \Lambda^-(\nu), \quad \nu \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right), \quad (\text{A } 34)$$

and

$$N^{(2)}(\nu) = \nu \Lambda^+(\nu) \Lambda^-(\nu), \quad \nu \in \left(-1, -\frac{1}{\sigma}\right) \cup \left(\frac{1}{\sigma}, 1\right) \quad (\text{A } 35)$$

Finally, the cross-product integrals are

$$\begin{aligned} N_{\alpha\beta}(\nu) &= \lambda_{a1\alpha}(\nu) \lambda_{1\beta}(\nu) + \lambda_{a2\alpha}(\nu) \lambda_{2\beta}(\nu) \\ &+ \pi^2 \nu^2 \left[\Delta_{a1\alpha}(\sigma\nu^2) \Delta_{1\beta}(\sigma\nu^2) + \Delta_{a2\alpha}(\nu^2) \Delta_{2\beta}(\nu^2) \right], \\ \alpha, \beta &= 1, 2, \quad \nu \in \left(-\frac{1}{\sigma}, \frac{1}{\sigma}\right) \end{aligned} \quad (\text{A } 36)$$

RESUMO

Neste trabalho são estabelecidas as funções adjuntas de meio-alcance ("half-range") concernentes as soluções elementares da equação de transporte uni-dimensional de dois grupos para espalhamentos linearmente anisotrópicos. A relação de ortogonalidade é baseada nas funções matriciais que podem ser calculadas a partir das equações do tipo Fredholm normais. Todas as integrais necessárias são avaliadas e diversos problemas típicos são resolvidos de maneira concisa.

A relação de ortogonalidade estabelecida é usada para reduzir problemas de reatores críticos do tipo placa infinita na forma conveniente para a computação. Os valores exatos das dimensões críticas são obtidos para um conjunto selecionado de parâmetros. São discutidos, também, os auto-valores e as matrizes fundamentais.

RÉSUMÉ

Établissement de fonctions adjointes demiporté concernant solutions élémentaires des équations de Transport unidimensionnel à deux groupes pour ralentissement linéaire anisotropique.

La relation d'orthogonalité demiporté est basée sur fonctions matricielles qu'on peut calculer moyennant équations régulières du type Fredholm. Toutes les intégrales nécessaires sont résolues d'une façon concise.

La relation d'orthogonalité établie est utilisée pour permettre le calcul simplifié des problèmes relatifs à plaque critique.

On obtient des valeurs exactes pour les dimensions critiques par groupes de paramètres sélectionnés. On présente aussi le calcul des valeurs propres et des matrices fondamentales.

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