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ABSTRACT

A new method to reduce multiregion problems in plane geometry to sets of regular integral equations for the coefficients of the singular-eigenfunction expansions is proposed. The method is based on the half-range orthogonality relations of the eigenfunctions and can be used to solve numerically any multiregion and multimedia problems. Several model problems, from half-space to two-slab problems, are solved and numerical results are reported. Some of the problems have been solved by other methods: for those problems available methods of solution are discussed. Others are solved here for the first time.

The basic theory is briefly reviewed and necessary formulas are summarized. Fundamental numerical techniques used in transport calculations are also discussed.

1 – INTRODUCTION

Since the publication of Case's fundamental paper$^{(5)}$ in 1960 the transport equation for plane geometry has been studied extensively in various models. The general solution of the transport equation is given as an expansion in eigenfunctions, some of which are singular, with arbitrary expansion coefficients. As Case demonstrated in his paper on one-group theory$^{(5)}$ and has been done in several other models$^{(12,13,15-17,19)}$, it is then proved that the solution is complete in the sence that the expansion, with appropriate coefficients, can represent an arbitrary function.

If we consider a specific problem, the question then becomes how to determine the expansion coefficients such that the solution satisfies the boundary and interface conditions imposed by physical considerations. These conditions, due to the very fact that some of the eigenfunctions are singular, result in a set of singular integral equations. To solve this set of singular integral equations various orthogonality relations of the eigenfunctions have been obtained: full-range, half-range, and two media orthogonality relations. Using these orthogonality relations several problems have been solved analytically. Some other problems were reduced to a set of regular integral equations for the coefficients and this set of equations was then solved numerically by iterations. For many problems, however, even this second kind of solution has not been possible.

In the one-group model for isotropic scattering, i.e., the simplest and most-studied model, the infinite-medium Green's functions problem, half-space problems, and two-half-space problems can be solved analytically using the full-range, half-range, and two-media orthogonality relations, respectively$^{(6)}$. Single-slab problems can be reduced to regular integral equations using the half-range orthogonality-relations. The two slab cell problem has been solved by Bond and Siewert$^{(2)}$ who removed the singularity by judiciously manipulating the equations. More recently a new method based on the half-range orthogonality relations and invariance principles$^{(9)}$ was introduced and used to solve some two-media problems by Siewert and Burkart$^{(18)}$. However, Bond and Siewert's methods does not apply to nonsymmetric problems and the extentionof the last method to multiregion problems are not straightforward. Thus a general systematic method to solve various problems has been lacking and many model problems have remained unsolvable.

The purpose of this report is to propose a general method to derive a set of regular integral equations for the coefficients of The singular-eigenfunction expansion for multiregion problems, that can be solved by a standard iterative method, from the set of singular integral equations that results from boundary and interface conditions and to report numerical results for some model problems.

The method requires the half-range orthogonality relations and the accompanying H functions, but neither the full-range nor the two-media orthogonality relations, and can be used to solve numerically any multiregion and multimedia problem in plane geometry.

We shall first review the basic theory briefly and summarized the half-range orthogonality relations, related formulas, and the method of regularization. Then we shall consider five geometries, from a half-space to two slabs, and solve one problem in each geometry. Extensions to problems involving three or more regions are straightforward and should be clear from these examples. Some of the problems have been solved by other methods: for those problems available methods of solution are discussed. Others are solved here for the first time. In the course of the work fundamental numerical techniques are also discussed.

Extensions to the case of anisotropic scattering is also considered: in Section 12 we summarize basic theory and necessary formulas and report some numerical results for the problem of neutron transmission through two slabs.

The one-group transport equation for an isotropically scattering homogeneous medium in plane geometry can be written as

$$\mu \frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu) = \frac{C}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu' \tag{1}$$

where $\psi(x,\mu)$ is the angular flux, x is the distance in units of the mean-free-path, and μ is the cosine of the angle between the direction of neutron motion and the *x* axis. *The parameter c is the average number of secondary neutrons per collision and is the sole parameter (besides the mean-free-path) required to characterize the transport of neutrons in a medium. It is assumed that boundary conditions are uniform and azimuthally symmetric.*

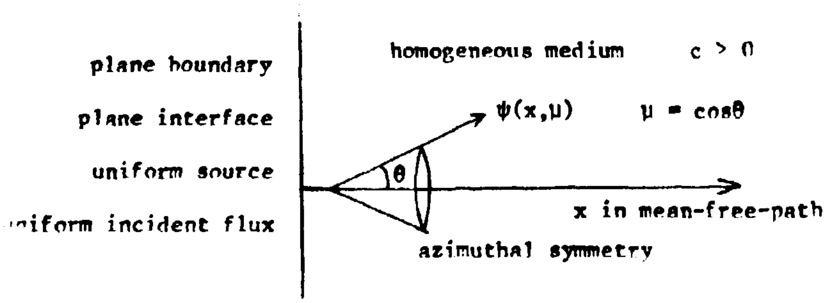

Figure 1 - The Plane Geometry

The solution of Eq.(1) is given by the expansion[6]

$$\psi(x,\mu) = A(\nu_o)\,\phi(\nu_o,\mu)\exp(-x/\nu_o) + A(-\nu_o)\,\phi(-\nu_o,\mu)\exp(x/\nu_o)$$

$$+ \int_{-1}^{1} A(\nu)\,\phi(\nu,\mu)\exp(-x/\nu)\,d\nu\,, \tag{2}$$

where the discrete eigenvalues $\pm\,\nu_0$ are the zeroes of the dispersion function

$$\Lambda(z) = 1 - \frac{c}{2}\,z\,\ln\frac{1+1/z}{1-1/z}\quad,\quad z \neq \text{real}\in(-1,1)\quad, \tag{3}$$

the discrete eigenfunctions are

$$\phi(\pm\nu_o,\mu) = \frac{c\nu_o}{2}\,\frac{1}{\nu_o\mp\mu}\quad, \tag{4a}$$

and for continuum eigenvalues $\nu\epsilon(-1,1)$ the eigenfunction is given as

$$\phi(\nu\ \mu) = \frac{c\nu}{2}\,\frac{P}{\nu-\mu} + \lambda(\nu)\,\delta(\nu-\mu) \tag{4b}$$

with

$$\lambda(\nu) = 1 - \frac{c}{2}\,\nu\,\ln\frac{1+\nu}{1-\nu} \tag{5}$$

The functions Λ and λ are shown in Figure 2 for some values of $c < 1$.

The eigenfunctions have the properties

$$\int_{-1}^{1}\phi(\xi,\mu)\,d\mu = 1 \tag{6}$$

and

$$\phi(\xi,-\mu) = \phi(-\xi,\mu)\quad,\quad \xi = \pm\nu_o \text{ or } \nu\epsilon(-1,1)\ . \tag{7}$$

The symbol P is used to denote the Cauchy principal-value integral[8]

$$\int_a^b \frac{P}{x-c}\,dx = \lim_{\epsilon\to 0}\left\{\int_a^{c-\epsilon} \frac{1}{x-c}\,dx + \int_{c+\epsilon}^{b} \frac{1}{x-c}\,dx\right\} = \ln\frac{b-c}{c-a}, \quad a<c<b \tag{8}$$

The general solution of a problem may contain, in addition to the expansion in Eq. (2), a particular solution. The coefficients $A(\pm\nu_0)$ and $A(\nu)$ must be determined such that the solution satisfies the boundary and other conditions. Is has been proved$^{(6)}$ that the expansion in Eq.(2) is complete and the eigen-functions are orthogonal in the full range $\mu \in (-1, 1)$ and in the half range $\mu \in (0,1)$.

It is known$^{(6)}$ that the dispersion function $\Lambda(z)$ has one pair of zeroes $\pm\nu_0$ in the entire complex plane cut along the real axis from -1 to 1 and thar ν_0 is real for $c < 1$ and pure imaginary for $c > 1$. The discrete eigenvalue ν_0 is known explicitly$^{(4)}$:

$$\nu_0^2 = -1 + \left(2 - \frac{c}{2}\pi\right)(1-c)^{-1} \exp\left\{-\frac{2}{\pi}\int_0^1 \tan^{-1}\left[\frac{c\pi\mu}{2\lambda(\mu)}\right] - \frac{\mu}{\mu^2+1}\,d\mu\right\} . \tag{9}$$

However, the integral in this formula is rather difficult to evaluate numerically and it is more accurate and expedient to use an iterative method to calculate ν_0, using the value obtained from Eq.(9) as an initial value.

Several valus of ν_0 are shown in Table I.

In the following we write the general solution, Eq.(2), of the transport equation in a slightly modified form as

$$\begin{aligned}\psi(x,\mu) &= A(\nu_0)\,\phi(\nu_0,\mu)\exp(-x/\nu_0) + A(-\nu_0,\mu)\,\phi(-\nu_0,\mu)\exp(x/\nu_0)\\ &+ \int_0^1 A(\nu)\,\phi(\nu,\mu)\exp(-x/\nu)\,d\nu + \int_0^1 A(-\nu)\,\phi(-\nu,\mu)\exp(x/\nu)\,d\nu\end{aligned} \tag{10}$$

and use the symbols ν_0, ν_i, ν and ξ to denote positive eigenvalues.

Table I

The Discrete Eigenvalue

c	ν_0	c	ν_0
0.4	1.014585815927	0.9	1.903204856045
0.5	1.044382033608	0.95	2.635148834269
0.6	1.102132021151	0.99	5.796729451302
0.7	1.206804253985	1.01	i 5.750539872535
0.8	1.407634309063	1.05	i 2.302327087032
0.85	1.588558625363	1.20	i 1.198265001513

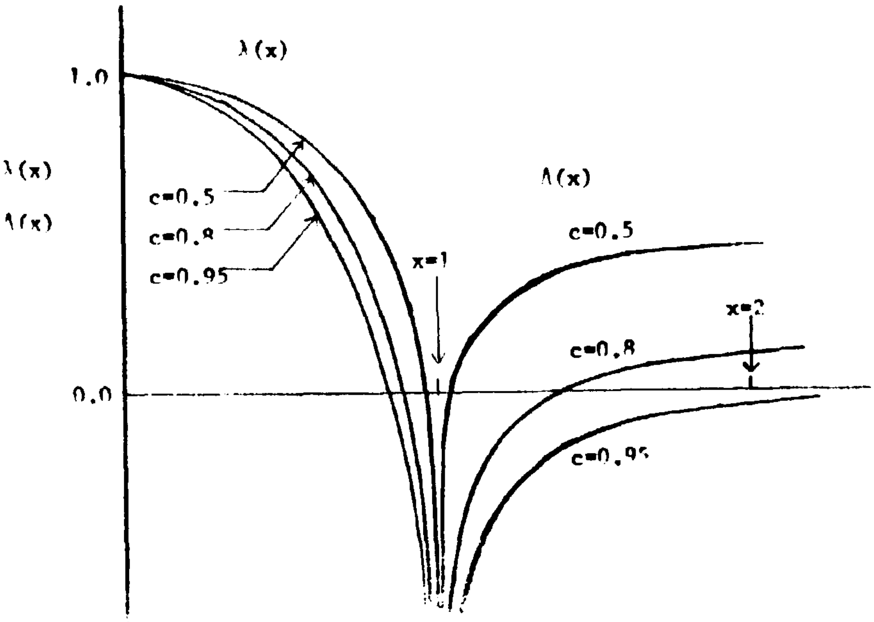

Figure 2 – The Functions λ and Λ

2 – HALF-RANGE COMPLETENESS AND ORTHOGONALITY

In some problems boundary conditions result in an equations of the form

$$A(\nu_0)\,\phi(\nu_0,\mu) + \int_0^1 A(\nu)\,\phi(\nu,\mu)\,d\nu = f(\mu) \quad , \qquad \mu\epsilon(0,1) \tag{11}$$

It has been proved[6] that the expansion in Eq.(11) is complete in the half range $\mu\,\epsilon(0,1)$ for arbitrary function $f(\mu)$ satisfying the Hölder condition

$$|f(\mu) - f(\mu')| < M\,|\mu - \mu'|^{\alpha} \quad , \tag{12}$$

where M is a positive constant and $0 < \alpha < 1$.

The eigenfunctions $\phi(\nu_0,\mu)$ and $\phi(\nu,\mu)$, $\nu\ \epsilon(0,1)$, are orthogonal in the half range $\mu\epsilon(0,1)$ in the sense that

$$\int_0^1 \mu H(\mu)\,\phi(\xi,\mu)\,\phi(\xi',\mu)\,d\mu = 0 \quad , \quad \xi \neq \xi' \ ; \quad \xi,\xi' = \nu_0 \text{ or } \nu\epsilon(0,1) \ . \tag{13}$$

Thus Eq.(11) can be solved for the coefficients to give

$$A(\nu_o) = \int_o^1 \mu H(\mu)\,\phi(\nu_o,\mu)\,f(\mu)\,d\mu \;/\; \int_o^1 \mu H(\mu)\,\phi(\nu_o,\mu)\,\phi(\nu_o,\mu)\,d\mu \qquad (14a)$$

and

$$A(\nu) = \int_o^1 \mu H(\mu)\,\phi(\nu,\mu)\,f(\mu)\,d\mu \;/\; \int_o^1 \mu H(\mu)\,\phi(\nu_o,\mu)\,\phi(\nu_o,\mu)\,d\mu \qquad (14b)$$

The function $H(\mu)$ was originally introduced by Chandraseckar$^{(7)}$ in regard to radiative transfer problems. Since in our work the H function and the half-range orthogonality relations play a principal role, we summarize their properties and related formulas in the following two sections.

3 – THE H FUNCTION

Though the H function can be defined in terms of the solution of a Reimann-Hilbert boundary-value problem related to the half-range completeness proof, it can be introduced in a more straightforward way through invariant independding analysis.

The Fundamental equation is a nonlinear integral equations

$$H(\mu) = 1 + \frac{c}{2}\,\mu H(\mu) \int_o^1 H(\mu')\,\frac{1}{\mu' + \mu}\,d\mu' \;, \quad \mu\epsilon(0,1) \;. \qquad (15)$$

The definition of the H finction can be extended to the entire complex plane cut along the real axis from −1 to 1 and Eq.(15) can be modified to give

$$H^{-1}(z) = 1 - \frac{c}{2}\,z \int_o^1 H(\mu)\,\frac{1}{\mu + z}\,d\mu \;, \quad z \neq \text{real}\ \epsilon(-1,0) \qquad (16)$$

The H function has a property such that

$$\int_{-1}^1 \phi(\xi,\mu)\,d\mu = \int_o^1 \phi(\xi,\mu)\,H(\mu)\,d\mu \;, \quad \xi = \nu_o \text{ or } \nu\epsilon(0,1) \;, \qquad (17)$$

and, therefore, satisfies the equations

$$\frac{c}{2}\,\nu_o \int_o^1 H(\mu)\,\frac{1}{\nu_o - \mu}\,d\mu = 1 \;, \qquad (18a)$$

and

$$\frac{c}{2}\nu\int_0^1 H(\mu)\frac{\mu}{\nu-\mu}d\mu = 1-\lambda(\nu)H(\nu) \quad , \quad \nu\in(0,1) \quad . \tag{18b}$$

From Eq.(15) we can derive a moment relation

$$cH_0^2 = 4(H_0-1) \tag{19a}$$

where the moment of the H function is defined by

$$H_\alpha = \int_0^1 H(\mu)\,\mu^\alpha\,d\mu \quad , \quad \alpha = \text{integer} \quad . \tag{19b}$$

The H function is known explicitly[18]:

$$H(\mu) = \frac{1+\mu}{(\nu_0+\mu)\sqrt{1-c}}\exp\left\{-\frac{1}{\pi}\int_0^1 \tan^{-1}\left[\frac{c\pi\mu'}{2\lambda(\mu')}\right]\frac{1}{\mu'+\mu}d\mu'\right\} \quad . \tag{20}$$

We can conclude from Eqs.(18a) and (20) that $H(\mu)$ is complex for $c > 1$, and from Eqs.(16) and (18a) that

$$H^{-1}(-\nu_0) = 0 \quad , \tag{21}$$

i.e., $H(z)$ is singular at $z=-\nu_0$.

The extended function $H(z)$ satisfies the equation

$$H(z)\;\Lambda(z) = 1+\frac{c}{2}z\int_0^1 H(\mu)\frac{1}{\mu-z}d\mu \tag{22}$$

from which Eq.(18b) can be obtained as a limiting case since

$$\lambda(\nu) = \frac{1}{2}\left\{\Lambda^+(\nu)+\Lambda^-(\nu)\right\} \quad , \quad \nu\in(-1,1) \tag{23}$$

From Eqs.(16) and (22) we obtain

$$H(z)\,H(-z)\,\Lambda(z) = 1 \tag{24}$$

and thus we have

$$H(\infty) = \{\Lambda(\infty)\}^{-1/2} = (1-c)^{-1/2} \; . \tag{25}$$

Numerical values of H(μ) can be obtained from Eq.(20). However, to achieve a good accuracy many quadrature points must be taken near $\mu = 1$. It is easier and more accurate to calculate H(μ) from Eq.(15) by iteration. Convergence is good for small valus of c and for $c \approx 1$ and $c > 1$ another equation is available( ):

$$H^{-1}(\mu) = \frac{\nu_0+\mu}{\nu_0(1+\mu)} - \frac{c}{2}\,\mu\,\frac{\nu_0+\mu}{1+\mu}\int_0^1 H(\mu')\,\frac{1-\mu'}{(\nu_0-\mu')(\mu'+\mu)}\,d\mu' \; . \tag{26a}$$

which is derived by combining Eqs.(15) and (18a) using the identity

$$\frac{1+\mu}{\nu_0+\mu} = \frac{1-\mu'}{\nu_0-\mu'} - \frac{(1-\nu_0)(\mu+\mu')}{(\nu_0-\mu')(\nu_0+\mu)} \; . \tag{26b}$$

The number of iterations required for the same accuracy using Eqs.(15) and (26a) is compared in Table II togheter with numerical examples of the H function.

Table II

The H Function

μ	c = 0.6	c = 0.8	c = 0.95
	H(μ)	H(μ)	H(μ)
0.0	1.0	1.0	1.0
0.1	1.09134859	1.13880767	1.19523181
0.2	1.14516451	1.22863877	1.33733678
0.3	1.18586787	1.30058828	1.46045391
0.4	1.21860018	1.36108970	1.57095414
0.5	1.24580716	1.41326257	1.67178829
0.6	1.26892012	1.45898949	1.76471161
0.7	1.28887095	1.49953974	1.85091741
0.8	1.30630761	1.53582791	1.93128623
0.9	1.32170140	1.56854278	2.00650369
1.0	1.33640667	1.59821952	2.07712384
α	H_α	H_α	H_α
0	1.22514823	1.38196601	1.63451200
1	0.63663257	0.73581523	0.90187602
2	0.43092070	0.50322379	0.62679444
3	0.32577495	0.38259512	0.48093912
Number of iterations for $e = 10^{-10}$			
Eq.(15)	10	14	26
Eq.(26)	9	11	11

4 – ORTHOGONALITY FORMULAS AND RELATED INTEGRALS

The half-range orthogonality relations of the eigenfunctions can be summarized as

$$\int_0^1 \mu H(\mu)\,\phi(\nu_0,\mu)\,\phi(\nu,\mu)\,d\mu = 0 \quad , \tag{27a}$$

$$\int_0^1 \mu H(\mu)\,\phi(\nu_0,\mu)\,\phi(\nu_0,\mu)\,d\mu = H(\nu_0)\,N(\nu_0) \quad . \tag{27b}$$

and

$$\int_0^1 \mu H(\mu)\,\phi(\nu,\mu)\,\phi(\nu',\mu)\,d\mu = H(\nu)\,N(\nu)\,\delta(\nu-\nu') \quad , \tag{27c}$$

where

$$N(\nu_0) = \frac{c}{2}\,\nu_0^3\,\left\{\frac{c}{\nu_0^2-1} - \frac{1}{\nu_0^2}\right\} \quad , \tag{28a}$$

and

$$N(\nu) = \nu\left\{\lambda(\nu)\,\lambda(\nu) + \frac{1}{4}\,\pi^2\,c^2\nu^2\right\} \quad . \tag{28b}$$

Since in our work we shall need various integrals involving products of eigenfunctions, we summarize them here. For eigenfunctions belonging to the same medium

$$\int_0^1 \mu H(\mu)\,\phi(\xi,\mu)\,\phi(-\xi',\mu)\,d\mu = \frac{c}{2}\,\frac{\xi\xi'}{\xi+\xi'}\,H^{-1}(\xi') \quad , \quad \xi,\xi' = \nu_0 \text{ or } \nu\epsilon(0,1) \tag{29}$$

For eigenfunctions belonging to different media, with $\xi_i = \nu_i$ or $\nu\epsilon(0,1)$,

$$\int_0^1 \mu H_i(\mu)\,\phi_i(\xi_i,\mu)\,\phi_j(-\xi_j,\mu)\,d\mu = \frac{c_j}{2}\,\frac{\xi_i\xi_j}{\xi_i+\xi_j}\,H_i^{-1}(\xi_i) \tag{30}$$

$$\int_0^1 \mu H_i(\mu)\,\phi_i(\xi_i,\mu)\,\phi_j(\nu_j,\mu)\,d\mu = \frac{c_j}{2}\,\frac{\nu_j\xi_i}{\nu_j-\xi_i}\,H_i^{-1}(-$$ [illegible] $$) \tag{31a}$$

$$\int_0^1 \mu H_i(\mu)\,\phi_i(\nu_i,\mu)\,\phi_j(\nu,\mu)\,d\mu = \frac{c_i-c_j}{2}\,\frac{\nu_i\nu}{\nu_j-\nu}\,H_i(\mu) \quad . \tag{31b}$$

and

$$\int_0^1 \mu H_i(\mu)\,\phi_i(\nu,\mu) \int_0^1 A_j(\nu')\,\phi_j(\nu',\mu)\,d\nu'\,d\mu$$

$$= \frac{c_i - c_j}{2}\,\nu \int_0^1 \nu' A_j(\nu')\,H_i(\nu')\,\frac{P}{\nu - \nu'}\,d\nu'$$

$$+ \nu A_j(\nu)\,H_j(\nu)\,\left\{ \lambda_i(\nu)\,\lambda_j(\nu) + \frac{c_i - c_j}{4}\,\pi^2\nu^2 \right\} \tag{31c}$$

Here we have used the subscripts to refer to the media and written the discrete eigenvalues by ν_i. In the last formula $A_j(\nu)$ is an arbitrary function satisfying the Holder condition. We not that the last integral is still singular and in fact this is the type of integral we shall be concerned with.

Finally the following recursion relation is of interest:

$$\int_0^1 \mu^{\alpha} H(\mu)\,\phi(\xi,\mu)\,d\mu = \xi \left\{ \int_0^1 \mu^{\alpha-1} H(\mu)\,\phi(\xi,\mu)\,d\mu - \frac{c}{2} H_{\alpha-1} \right\} \quad , \quad \alpha = 1, 2, \ldots, \tag{32a}$$

with

$$\int_0^1 H(\mu)\,\phi(\xi,\mu)\,d\mu = 1 \tag{32b}$$

5 – THE METHOD OF REGULARIZATION

The method to regularize singular integrals of the type of Eq.(31c) can be summarized in the following steps:

1) At an interface separate the continuity into two equations, one for $\mu \epsilon (0,1)$, the other for $\mu \epsilon (-1,0)$.

2a) To the $\mu \epsilon (0,1)$ equation apply the half-range orthogonality relations for the right-side medium.

b) In the $\mu \epsilon (-1,0)$ equation change μ to $-\mu$ and then apply the orthogonality relations for the left-side medium.

3a) If any singularity remains in step 2a, consider the interface (or boundary) condition for $\mu > 0$ at the left-side boundary of the left-side medium and generate the same singularity, subtract the result from the equation in step 2a and remove the singularity.

b) For step consider the right-side interface of the right-side medium and generate the same singularity from the $\mu < 0$ equation.

4) If singularities remain in step 3 repeat the process, generating the same singularities at different interfaces.

Although the equation for a discrete coefficient is always found to be regular, we apply to this equation the same operations as those applied to the equation for the corresponding continuum coefficient, since the convergence of iterations is sometimes faster and the discrete and continuum coefficients are obtained in the form. We note that for a symmetric geometry the right and left interfaces are equivalent.

6 – QUADRATURE FORMULAS AND ITERATIVE SOLUTION OF INTEGRAL EQUATIONS

A Gaussian quadrature set$^{(1)}$ can be used to numerically evaluate an integral in the form

$$\int_{-1}^{1} f(y)\,dy = \sum_{i=1}^{N} f(y_i)\,u_i \tag{33}$$

where N is the order, y_i the nodes, and u_i the weights of the quadrature set. This formula is exact if the function f is a polynomial of order 2N – 1 or less. The nodes and weights are found in Ref. .

If the interval of integration is different from (-1,1), the nodes and weights can be modified to give

$$\int_{a}^{b} f(x)\,dx = \sum_{i=1}^{N} f(x_i)\,w_i \tag{34}$$

with

$$x_i = \frac{b-a}{2} y_i + \frac{b+a}{2} \tag{35a}$$

and

$$w_i = \frac{b-a}{2} u_i \ . \tag{35b}$$

An integral equation of the form

$$F(x) = f(x) + \int_{a}^{b} K(x,y)\,F(y)\,dy \ , \quad x \epsilon (a,b) \ , \tag{36}$$

where f(x) and K(x,y) are known functions, can be solved numerically in the following iterative scheme

$$F^{(\alpha)}(x_i) = f(x_i) + \sum_{j=1}^{i-1} K(x_i,y_j)\,F^{(\alpha)}(y_j)\,w_j$$

$$+ \sum_{j=1}^{N} K(x_i,y_j)\,F^{(\alpha-1)}(y_j)\,w_j \ , \quad i = 1, 2, \ldots, N \ , \tag{37}$$

where $F^{(\alpha)}(x_i)$ is the value of F at x_i after the α-th iteration. The iteration can be terminated when

$$|F^{(\alpha)}(x_i) - F^{(\alpha-1)}(x_i)| < \epsilon \qquad \text{for all } i \ , \tag{38}$$

where ϵ is the desired accuracy.

7 – THE MILNE PROBLEM FOR A HALF-SPACE: PROBLEM 1

The Milne problem for a half-space is probably the best-known model problem in transport theory. The problem originated in the study of radiative transfer as a model of the sun (or starts), but it has a relation to nuclear engineering through the formula for the extrapolated endpoint that can be used in boundary conditions in diffusion theory or in that it can be considered as a model of a thermal column. There are other half-space problems often considered in transport theory, e.g., the albedo and constant-source problems. However, since these problems are mathematically equivalent, we consider here only the Milne problem.

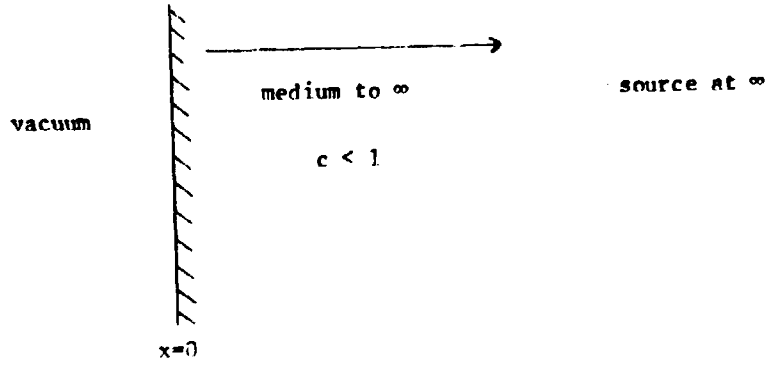

Figure 3 — Problem 1

The solution can be written as

$$\psi(x,\mu) = A(\nu_0)\,\phi(\nu_0,\mu)\exp(-x/\nu_0) + \phi(-\nu_0,\mu)\exp(x/\nu_0)$$

$$+ \int_0^1 A(\nu)\,\phi(\nu,\mu)\exp(-x/\nu)\,d\nu \ , \quad x > 0. \ . \tag{39}$$

The coefficient $A(-\nu)$ in the general solution, Eq.(10), has been taken to be zero by the condition for $x \to \infty$. The coefficient $A(-\nu_0)$ may be considered to have been normalized to unity, or the term $\phi(-\nu_0,\mu)\exp(x/\nu_0)$ may be considered a particular solution with $A(-\nu_0) = 0$. The remaining boundary condition $\psi(0,\mu) = 0$, $\mu\epsilon(0,1)$, results in the equation

$$A(\nu_0)\,\phi(\nu_0,\mu) + \int_0^1 A(\nu)\,\phi(\nu,\mu)\,d\nu = -\phi(-\nu_0,\mu) \quad , \quad \mu\epsilon(0,1) \quad . \tag{40}$$

If we apply the half-range orthogonality theorem, i.e., multiply Eq.(40) by $\mu H(\mu)\phi(\xi,\mu)$, $\xi=\nu_0$ or $\nu\epsilon(0,1)$, and integrate over $\mu\epsilon(0,1)$, we obtain, using Eqs.(27) and (29),

$$A(\xi) = -\xi N^{-1}(\xi) H^{-1}(\xi)\,\frac{c}{2}\,\frac{\nu_0}{\nu_0+\xi}\,H^{-1}(\nu_0) \tag{41}$$

The total (scalar) flux is defined by

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,d\mu \tag{42}$$

and thus, recalling Eq.(6), we can write

$$\phi(x) = A(\nu_0)\exp(-x/\nu_0) + \exp(x/\nu_0) + \int_0^1 A(\nu)\exp(-x/\nu)\,d\nu \quad . \tag{43}$$

The extrapolated endpoint has been calculated in various models as a convenient measure of the asymptotic behaviour of the total flux and is defined by the zero-extrapolation of the asymptotic flux,

$$A(\nu_0)\exp(z_0/\nu_0) + \exp(-z_0/\nu_0) = 0 \quad , \tag{44}$$

which can be solved to give

$$z_0 = -\frac{\nu_0}{2}\ln\{-A(\nu_0)\} \quad . \tag{45}$$

Numerical examples are given in Tables III and IV and in Figure 4 for several values of c.

With the coefficients known, the angular flux $\psi(x,\mu)$ can, of course, be calculated for any x and μ. The principal-value integral that appears in the continuum term can be evaluated in the following way:

$$\int_0^1 A(\nu)\,\frac{P}{\nu-\mu}\,d\nu = \int_0^1 \{A(\nu)-A(\mu)\}\,\frac{1}{\nu-\mu}\,d\nu + A(\mu)\int_0^1 \frac{P}{\nu-\mu}\,d\nu$$

$$= \int_0^1 \{A(\nu)-A(\mu)\}\,\frac{1}{\nu-\mu}\,d\nu + A(\mu)\ln\frac{1-\mu}{\mu} \quad . \tag{46}$$

The last integral is now regular and, since $A(\mu)$ is known for all μ, it can be evaluated by the standard quadrature. Though Eq.(46) is matematically exact, the accuracy of the calculated angular flux not be good for $\mu \simeq 1$. This is due to the behavior of $A(\nu)$ and the functions $\lambda(\nu)$ and $\ln\{(1-\mu)/\mu\}$ in this region, see Figures 2 and 4. To calculate the angular flux near $\mu = 1$ to a good accuracy, many quadrature points must be used in this region. Further, in Eq.(46) μ must not be equal to any of the nodal points.

Table III

Problem 1, Discrete Coefficient and Extrapolated Endpoint

c	$A(\nu_0)$	z_0
0.4	−0.027397	1.82490
0.5	−0.063340	1.44085
0.6	−0.114916	1.19226
0.7	−0.185037	1.01806
0.8	−0.282751	0.889055
0.85	−0.348923	0.836300
0.90	−0.436169	0.789569
0.95	−0.566874	0.747879
0.99	−0.780675	0.717624

Table IV

Problem 1, Exact and Asymptotic Total Flux

x	c = 0.5		c = 0.8		c = 0.95	
	$\phi(x)$	$R(x)^*$	$\phi(x)$	$R(x)^*$	$\phi(x)$	$R(x)^*$
0.0	0.73915	1.26721	0.48400	1.48193	0.26764	1.61831
0.1	0.90421	1.15341	0.64779	1.25081	0.37809	1.30368
0.2	1.05310	1.10034	0.78494	1.15597	0.46734	1.18416
0.4	1.35926	1.04726	1.04337	1.06946	0.62666	1.08013
0.6	1.70005	1.02385	1.30274	1.03388	0.77438	1.03858
0.8	2.09500	1.01275	1.57755	1.01749	0.91830	1.01957
1.0	2.56230	1.00725	1.87820	1.00941	1.06283	1.01020
1.5	4.17994	1.00237	2.79945	1.00206	1.44328	1.00193
2.0	6.77328	1.00065	4.07106	1.00029	1.87019	1.00027
3.0	17.6768	1.00006	8.39145	1.00003	2.94026	1.00004

* $R(x) = \phi_{asy}(x)/\phi(x)$

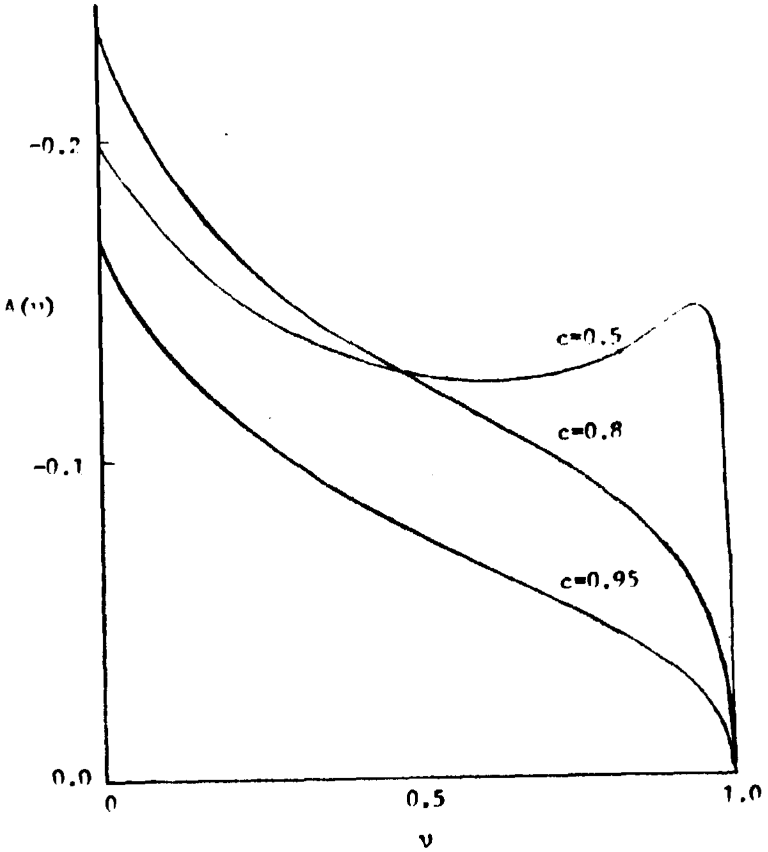

Figure 4 — Problem 1, Continuum Coefficient

8 — SINGLE SLAB WITH AN INCIDENT FLUX: PROBLEM 2

Probably the most often considered among single-slab problems is the critical problem for bare slab reactors(14). We consider here, however, a slab of thickness *a* irradiated from one side by a flux of neutrons $f(\mu)$.

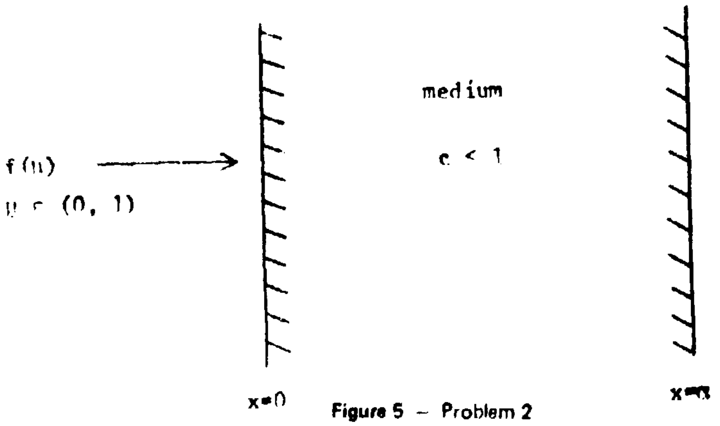

Figure 5 — Problem 2

The solution can be written as

$$\psi(x,\mu) = A(\nu_0)\,\phi(\nu_0,\mu)\exp(-x/\nu_0) + A(-\nu_0)\,\phi(-\nu_0,\mu)\exp\{-(\alpha-x)/\nu_0\}$$

$$+\int_0^1 A(\nu)\,\phi(\nu,\mu)\exp(-x/\nu)\,d\nu + \int_0^1 A(-\nu)\,\phi(-\nu,\mu)\exp\{-(\alpha-x)/\nu\}\,d\nu \ ,$$

$$0 \leq x \leq \alpha \ . \tag{47}$$

with boundary conditions given by

$$\psi(0,\mu) = f(\mu) \quad , \quad \mu\epsilon(0,1) \ . \tag{48a}$$

and

$$\psi(\alpha,\mu) = 0 \quad , \quad \mu\epsilon(-1,0) \tag{48b}$$

Applying the first boundary condition to the solution we have

$$A(\nu_0)\,\phi\,(\nu_0,\mu) + \int_0^1 A(\nu)\,\phi\,(\nu,\mu)\,d\nu = f(\mu) - A(-\nu_0)\,\phi\,(-\nu_0,\mu)\,E(\nu_0)$$

$$-\int_0^1 A(-\nu)\,\phi\,(-\nu,\mu)\,E\,(\nu)\,d\nu \quad , \quad \mu\epsilon(0,1) \ . \tag{49a}$$

and the other condition results, after changing μ to $-\mu$, in the equation

$$A(-\nu_0)\,\phi\,(\nu_0,\mu) + \int_0^1 A(-\nu)\,\phi\,(\nu,\mu)\,d\nu = -A(\nu_0)\,\phi\,(-\nu_0,\mu)\,E\,(\nu_0)$$

$$-\int_0^1 A(\nu)\,\phi\,(-\nu,\mu)\,E\,(\nu)\,d\nu \quad , \quad \mu\epsilon(0,1) \ , \tag{49b}$$

where $E(\xi) = \exp(-\alpha/\xi)$.

Equations (49) are coupled singular integral equations for the expansion coefficients $A(\pm\,\nu_0)$ and $A(\pm\nu)$. Analytical solutions of this set of equations are not known. However we can convert these equations into a set of regular integral equations that can be solved numerically by iteration. We consider the left sides of Eqs.(49) as half-range expansions and apply the half-range orthogonality theorem to isolate the coefficients to obtain

$$A(\xi) = A^0\,(\xi) - \xi N^{-1}\,(\xi) H^{-1}\,(\xi) Y_1\,(\xi) \ . \tag{50a}$$

and

$$A(\xi) = \xi N^{-1}(\xi) H^{-1}(\xi) Y_2(\xi) \tag{50b}$$

for $\xi = \nu_0$ or $\nu \epsilon (0,1)$ where

$$A^0(\xi) = \int_0^1 \mu H(\mu)\, \phi(\xi,\mu)\, f(\mu)\, d\mu \tag{51}$$

$$Y_1(\xi) = A(\nu_0)\, \frac{c}{2}\, \frac{\nu_0}{\nu_0+\xi}\, H^{-1}(\nu_0)\, E(\nu_0) + \int_0^1 A(\nu)\, \frac{c}{2}\, \frac{\nu}{\nu+\xi}\, H^{-1}(\nu)\, E(\nu)\, d\nu \;, \tag{52}$$

and

$$Y_2(\xi) = A(\nu_0)\, \frac{c}{2}\, \frac{\nu_0}{\nu_0+\xi}\, H^{-1}(\nu_0)\, E(\nu_0) + \int_0^1 A(\nu)\, \frac{c}{2}\, \frac{\nu}{\nu+\xi}\, H^{-1}(\nu)\, E(\nu)\, d\nu \tag{53}$$

We consider here three cases of incident flux

$$f(\mu) = 2 \;,\quad \mu\epsilon(0,1) \;,\quad \text{Case 1} \;, \tag{54a}$$

$$f(\mu) = 3\mu \;,\quad \mu\epsilon(0,1) \;,\quad \text{Case 2} \;, \tag{54b}$$

$$f(\mu) = 4\mu^2 \;,\quad \mu\epsilon(0,1) \;,\quad \text{Case 3} \tag{54c}$$

The integral in Eq.(51) can be performed to give

$$A^0(\xi) = 2\xi N^{-1}(\xi) H^{-1}(\xi)\, \{1 - \frac{c}{2} H_0\} \;,\quad \text{Case 1} \;, \tag{55a}$$

$$A^0(\xi) = 3\xi N^{-1}(\xi) H^{-1}(\xi)\, \{\xi - \frac{c}{2}(\xi H_0 + H_1)\} \;,\quad \text{Case 2} \;, \tag{55b}$$

and

$$A^0(\xi) = 4\xi N^{-1}(\xi) H^{-1}(\xi)\, \{\xi^2 - \frac{c}{2}(\xi^2 H_0 + \xi H_1 + H_2)\} \;,\quad \text{Case 3} \;. \tag{55c}$$

The results of iteration can be checked in two ways for accuracy, i.e., how accurately the boundary-conditions are satisfied.

Firstly, Eqs.(48) can be checked pointwise for arbitrary values of μ. Though the equations contain a principal-value integral, it can be converted to a regular integral and a principal-value integral that can be integrated analytically by the technique used in the calculation of the angular flux, see Eq.(46). Secondly, Eqs.(48) can be multiplied by μ^α, $\alpha = 1, 2, \cdots$, or any other function, and integrated over $\mu\epsilon(0,1)$. Since the solution is exact, the results must be an identity for any α, i.e., from Eq.(48a) we must have

$$\int_0^1 \mu^\alpha \psi(0,\mu)\, d\mu = \int_0^1 \mu^\alpha f(\mu)\, d\mu \quad , \quad \alpha = 1, 2, \cdots \quad . \tag{56}$$

We shall use the terms "pointwise check" for the first kind and "moment check" for the second. For $\mu \simeq 1$ the pointwise check may not give good results. However, in the moment check large values of μ will have increasingly more weight as α increases. Thus the two checks combined should give sufficient confidence in the solutions.

The accuracy of iterative solutions depends on the quadrature set and the value of c. To obtain accurate results many quadrature points must be used near $\nu = 1$. The reason appears to be in that the continuum coefficients are not smooth in this region due to the fact that the normalization factor $N(\nu)$ diverges as $\nu \to 1$, see Figures 2 and 4. Table V shows the number of iterations required for $\epsilon = 10^{-8}$ and the accuracy, in the number of significant figures, that can be achieved with various quadrature sets. An approximate computation time required for one case is also shown. The notation $Q = N_1\epsilon(0,a) + N_2\epsilon(a,1)$ is used to denote that a N_1-point and a N_2-point Gaussian quadrature set is used in each of the intervals (0,a) and (a,1), respectively.

We report the total flux in Figure 6 and in Table VI the transmission rate defined as

$$\gamma = \int_0^1 \psi(\alpha,\mu)\mu\, d\mu \Big/ \int_0^1 f(\mu)\mu\, d\mu \quad . \tag{57}$$

Table V

Problem 2, Number of Iterations and Accuracy*

c	Iteration	Accuracy[a]	Accuracy[b]	Accuracy[c]	Accuracy[d]
0.5	4	4	5	6	7
0.8	6	5	6	6	7
0.95	11	6	7	7	7
Computation time		13	18	18	25

* Accuracy is given in the number of significant figures

* Computation time is the CPU time, in seconds, for c = 0.8

a $Q = 20\ \epsilon\ (0,1)$

b $Q = 40\ \epsilon\ (0\ 1)$

c $Q = 20\ \epsilon\ (0,0.99) + 20\ \epsilon\ (0.99,1)$

d $Q = 32\ \epsilon\ (0,0.99) + 32\ \epsilon\ (0.99,1)$

Table VI

Problem 2, Transmission Rate

c	α	Case 1	Case 2	Case 3
0.5	1	0.306709	0.344470	0.366916
0.5	2	0.107065	0.124591	0.136219
0.8	1	0.416245	0.451621	0.472063
0.8	2	0.197270	0.218042	0.231130
0.95	1	0.510978	0.543774	0.562213
0.95	2	0.3117461	0.341017	0.355135

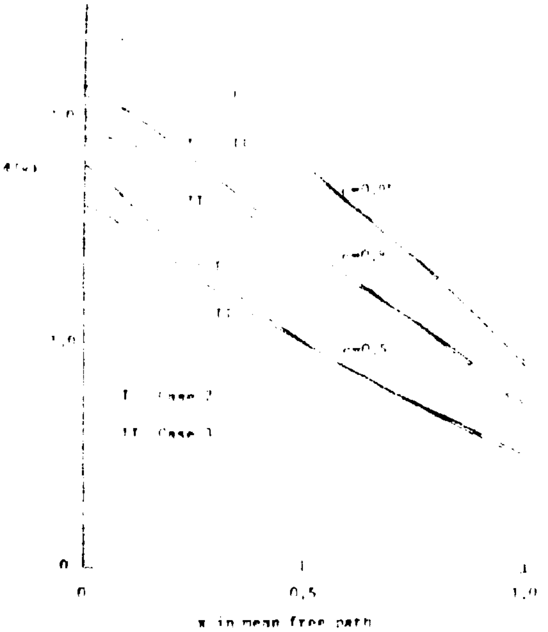

Figure 6 — Problem 2, The Scalar Flux

9 – THE MILNE PROBLEM FOR TWO HALF-SPACES: PROBLEM 3

As in the case of a single half-space, several problems can be considered in the same geometry, but we consider here only one of them, i.e., the Milne problem for two half-spaces of different medi

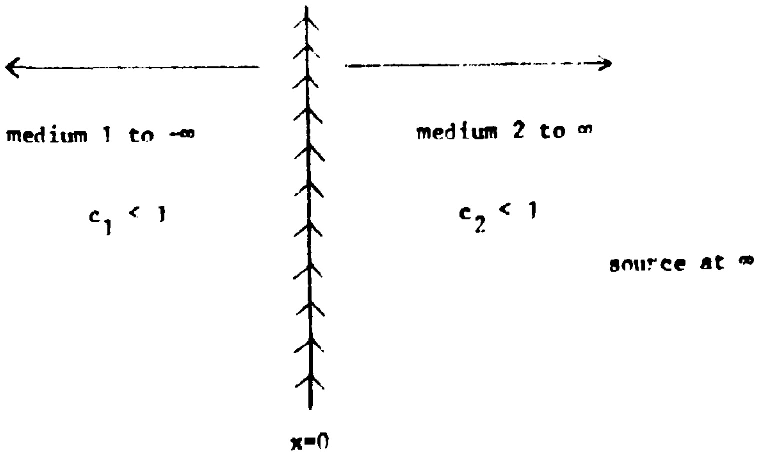

Figure 7 — Problem 3

The solutions can be written as

$$\psi_1(x,\mu) = A_1(-\nu_1)\,\phi_1(-\nu_1,\mu)\exp(x/\nu_1) + \int_0^1 A_1(-\nu)\,\phi_1(-\nu,\mu)\exp(x/\nu)\,d\nu \quad , \quad x \leq 0 \quad , \tag{58}$$

and

$$\psi_2(x,\mu) = A_2(\nu_2)\,\phi_2(\nu_2,\mu)\exp(-x/\nu_2) + \phi_2(-\nu_2,\mu)\exp(x/\nu_2)$$

$$+ \int_0^1 A_2(\nu)\,\phi_2(\nu,\mu)\exp(-x/\nu)\,d\nu \quad , \quad x \geqslant 0 \quad . \tag{59}$$

The interface condition $\psi_1(0,\mu) = \psi_2(0,\mu)$, $\mu\epsilon(-1,1)$, can be written in the two equations

$$A_1(-\nu_1)\,\phi_1(\nu_1,\mu) + \int_0^1 A_1(-\nu)\,\phi_1(\nu,\mu)\,d\nu = \phi_2(\nu_2,\mu) + A_2(\nu_2)\,\phi_2(-\nu_2,\mu)$$

$$+ \int_0^1 A_2(\nu)\,\phi_2(-\nu,\mu)\,d\nu \quad , \quad \mu\epsilon(0,1) \quad , \tag{60}$$

and

$$A_2(\nu_2)\,\phi_2(\nu_2,\mu) + \int_0^1 A_2(\nu)\,\phi_2(\nu,\mu)\,d\nu = -\phi_2(-\nu_2,\mu) + A_1(-\nu_1)\,\phi_1(-\nu_1,\mu)$$

$$+ \int_0^1 A_1(-\nu)\,\phi_1(-\nu,\mu)\,d\mu \quad , \quad \mu\epsilon(0,1) \quad . \tag{61}$$

If we apply the half-range orthogonality theorem for medium 1 to Eq.(60) and that for medium 2 to Eq.(61), we obtain a set of regular integral equations for the coefficients that can be solved by numerical iterations

$$A_1(-\xi. = A_1^0(-\xi) + \xi N_1^{-1}(\xi)H_1^{-1}(\xi)Y_1(\xi) \tag{62a}$$

and

$$A_2(\xi) = A_2^0(\xi) + \xi N_2^{-1}(\xi)H_2^{-1}(\xi)Y_2(\xi) \tag{62b}$$

where

$$A_1^0(\xi) = \xi N_1^{-1}(\xi)H_1^{-1}(\xi)\frac{c_2}{2}\,\frac{\nu_2}{\nu_2-\xi}\,H_1^{-1}(-\nu_2) \quad , \tag{63a}$$

$$A_2^0(\xi) = -\xi N_2^{-1}(\xi)H_2^{-1}(\xi)\frac{c_2}{2}\,\frac{\nu_2}{\nu_2+\xi}\,H_2^{-1}(\nu_2) \quad , \tag{63b}$$

$$Y_1(\xi) = A_2(\nu_2)\frac{c_2}{2}\frac{\nu_2}{\nu_2+\xi}H_1^{-1}(\nu_2) + \int_0^1 A_2(\nu)\frac{c_2}{2}\frac{\nu}{\nu+\xi}H_1^{-1}(\nu)\,d\nu \quad . \tag{63c}$$

and

$$Y_2(\xi) = A_1(-\nu_1)\frac{c_1}{2}\frac{\nu_1}{\nu_1+\xi}H_2^{-1}(\nu_1) + \int_0^1 A_1(-\nu)\frac{c_1}{2}\frac{\nu}{\nu+\xi}H_2^{-1}(\nu)\,d\nu \quad . \tag{63d}$$

We not that if $c_1 \to 0$ Eq.(62b) reduces to Eq.(41), the case of a singular half-space.

As was mentioned previously, the problem considered here can be solved explicitly using the two-media orthogonality relations or by the method used by Siewert and Burkart(18):

$$A_1(-\xi) = \xi N_1^{-1}(\xi)\frac{c_2}{2}\frac{\nu_2}{\nu_2+\xi}H_1^{-1}(\xi)H_2(\xi)H_1^{-1}(-\nu_2)H_2^{-1}(\nu_2) \quad , \tag{64a}$$

$$A_2(\xi) = -\xi N_2^{-1}(\xi)\frac{c_2}{2}\frac{\nu_2}{\nu_2+\xi}H_1(\xi)H_2^{-1}(\xi)H_1^{-1}(-\nu_2)H_2^{-1}(\nu_2) \quad . \tag{64b}$$

The extraporated endpoint z_0 is defined by the zero-extraporation of the asymptotic total flux in medium 2 and is given by

$$z_0 = -\frac{\nu_2}{2}\ln\{A_2(\nu_2)\} \quad . \tag{65}$$

We report our results, based on Eqs.(62), in Table VII and Figure 8 for several cases. We not that if $c_1 > c_2$, $A_2(\nu_2) > 0$ and z_0 cannot be defined. This point will be discussed in more detail in the next section.

Table VII

Problem 3, The Discrete Coefficients and the Extrapolated Endpoint

c_1	c_2	$A_1(-\nu_1)$	$A_2(\nu_2)$	z_0
0.9	0.95	0.800250	−0.162517	2.393999
0.8	0.95	0.590606	−0.311158	1.538209
0.7	0.95	0.454169	−0.388403	1.246045
0.5	0.95	0.234035	−0.473660	0.984548
0.95	0.8	1.488052	+0.286836	−
0.7	0.8	0.805699	−0.803962	1.774174
0.6	0.8	0.629046	−0.134362	1.412713
0.5	0.8	0.449886	−0.174027	1.230655
0.6	0.7	0.800065	−0.048856	1.821595
0.5	0.7	0.588255	−0.084995	1.487485

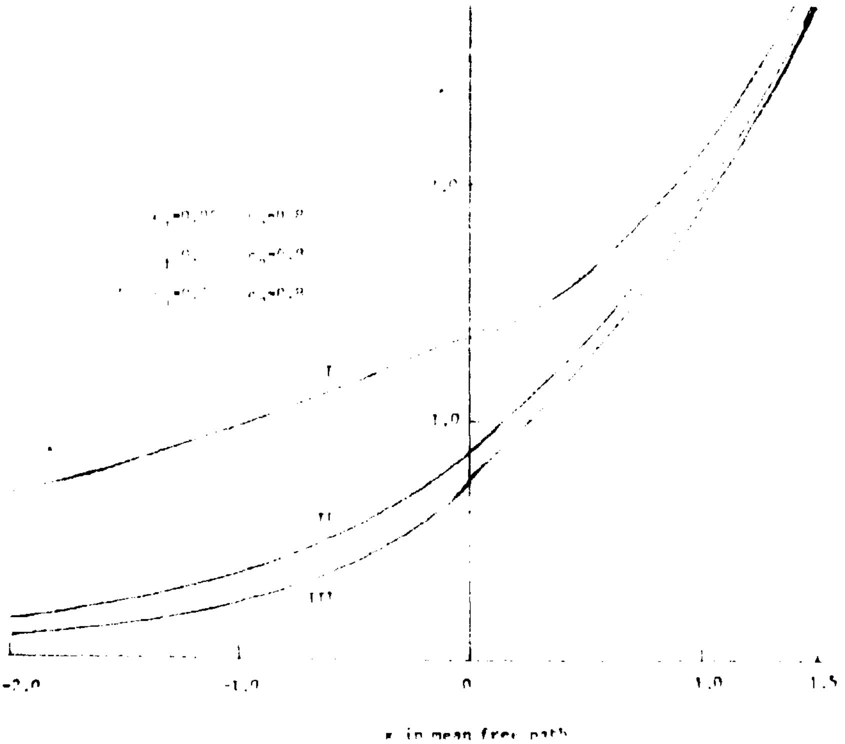

Figure 8 — Problem 3, The Scalar Flux

10 – THE MILNE PROBLEM FOR A HALF-SPACE BOUNDED BY A SLAB: PROBLEM 4

We consider here a problem similar to Problem 3: the Milne problem involving two media. However, we consider one of the media to a slab of finite thickness α. There is a plane source of neutrons deep in medium 2 $(x \geq 0)$ which is bounded at $x = 0$ by a slab of thickness α of medium 1 $(-\alpha \leq x \leq 0)$.

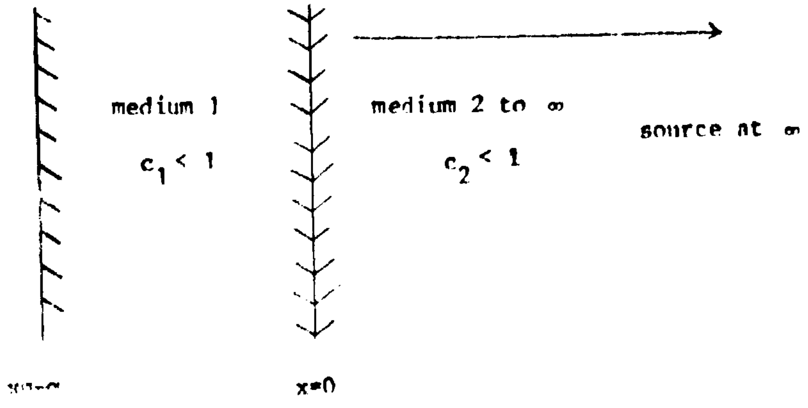

Figure 9 – Problem 4

The problem cannot be solved analytically nor have numerical been reported. The difficulty is in that the application of half-range orthogonality relations to the boundary condition at $x = -\alpha$ and full-range or two media orthogonality relations to the interface condition does not remove all singularities.

The method recently used by Siewert and Buckart[18] can be applied to this problem to derive a set or regular integral equations for the expanssion coefficients. We continue, however, to apply our method of regularization to this problem.

The solutions can be written as

$$\psi_1(x,\mu) = A_1(\nu_1)\,\phi_1(\nu_1,\mu)\exp\{-(x+\alpha)/\nu_1\} + A_1(-\nu_1)\,\phi_1(-\nu_1,\mu)\exp(x/\nu_1)$$

$$+ \int_0^1 A_1(\nu)\,\phi_1(\nu,\mu)\exp\{-(x+\alpha)/\nu\}\,d\nu + \int_0^1 A_1(-\nu)\,\phi_1(-\nu,\mu)\exp(x/\nu)\,d\nu \;,$$

$$\alpha \leqslant x \leqslant 0 \;, \tag{66a}$$

and

$$\psi_2(x,\mu) = A_2(\nu_2)\,\phi_2(\nu_2,\mu)\exp(-x/\nu_2) + \phi_2(-\nu_2,\mu)\exp(x/\nu_2)$$

$$+ \int_0^1 A_2(\nu)\,\phi_2(\nu,\mu)\exp(-x/\nu)\,d\nu \qquad x \geqslant 0 \;. \tag{66b}$$

The boundary condition at $x = -\alpha$ results in the equation

$$A_1(\nu_1)\,\phi_1(\nu_1,\mu) + \int_0^1 A_1(\nu)\,\phi_1(\nu,\mu)\,d\nu = -A_1(-\nu_1)\,\phi_1(-\nu_1,\mu)\,E(\nu_1)$$

$$- \int_0^1 A_1(-\nu)\,\phi_1(-\nu,\mu)\,E(\nu)\,d\nu \;, \quad \mu \in (0,1) \tag{67}$$

where $E(\xi) = \exp(-\alpha/\xi)$, and we write the interface condition in two equations

$$A_1(-\nu_1)\,\phi_1(\nu_1,\mu) + \int_0^1 A_1(-\nu)\,\phi_1(\nu,\mu)\,d\nu = \phi_2(\nu_2,\mu) + A_2(\nu_2)\,\phi_2(-\nu_2,\mu)$$

$$+ \int_0^1 A_2(\nu)\,\phi_2(-\nu,\mu)\,d\nu - A_1(\nu_1)\,\phi_1(-\nu_1,\mu)\,E(\nu_1)$$

$$- \int_0^1 A_1(\nu)\,\phi_1(-\nu,\mu)\,E(\nu)\,d\nu \;, \quad \mu \in (0,1) \;, \tag{68a}$$

and

$$A_2(\nu_2)\,\phi_2(\nu_2,\mu) + \int_0^1 A_2(\nu)\,\phi_2(\nu,\mu)\,d\nu = \phi_2(-\nu_2,\mu) + A_1(\nu_1)\,\psi_1(\nu_1,\mu)\,E(\nu_1)$$

$$+ A_1(\nu_1)\,\phi_1(\nu_1,\mu) + \int_0^1 A_1(\nu)\,\psi_1(\nu,\mu)\,E(\nu)\,d\nu$$

$$+ \int_0^1 A_1(\nu)\,\phi_1(\nu,\mu)\,d\nu \quad , \quad \mu \in (0,1) \quad . \tag{68b}$$

Applying the orthogonality relations for medium 1 to Eqs.(67) and (68a), we obtain regular equations

$$A_1(\xi) = \xi N_1^{-1}\,H_1^{-1}(\xi)Y_1(\xi) \quad , \tag{69}$$

and

$$A_1(-\xi) = A_1^0(-\xi) + \xi N_1^{-1}(\xi)H_1^{-1}(\xi)Y_2(\xi) \tag{70}$$

where

$$A_1^0(-\xi) = \xi N_1^{-1}(\xi)\,\frac{c_2}{2}\,\frac{\nu_2}{\nu_2-\xi}\,H_1^{-1}(\xi)H_1^{-1}(-\nu_2) \quad , \tag{71a}$$

$$Y_1(\xi) = A_1(-\nu_1)\,\frac{c_2}{2}\,\frac{\nu_1}{\nu_2+\xi}\,H_1^{-1}(\nu_1)\,E(\nu_1) + \int_0^1 A_1(-\nu)\,\frac{c_1}{2}\,\frac{\nu}{\nu+\xi}\,H_1^{-1}(\nu)\,E(\nu)\,d\nu \quad , \tag{71b}$$

and

$$Y_2(\xi) = -A_1(\nu_1)\,\frac{c_1}{2}\,\frac{\nu_1}{\nu_1+\xi}\,H_1^{-1}(\nu_1)\,E_1(\nu_1) - \int_0^1 A_1(\nu)\,\frac{c_1}{2}\,\frac{\nu}{\nu+\xi}\,H_1^{-1}(\nu)\,E(\nu)\,d\nu$$

$$+ A_2(\nu_2)\,\frac{c_2}{2}\,\frac{\nu_2}{\nu_2+\xi}\,H_1^{-1}(\nu_2) + \int_0^1 A_2(\nu)\,\frac{c_2}{2}\,\frac{\nu}{\nu+\xi}\,H_1^{-1}(\nu)\,d\nu \tag{71c}$$

If we apply the orthogonality relations for medium 2 to Eq.(68b), we can isolate the coefficients in the left side. However in the equation for $A_2(\nu)$, the $A_1(\nu)$ term remains singular, involving a singular integral of the type that appears in Eq.(31c). Following the steps of regularization given in Section 5, we next multiply Eq.(67) by $\mu H_2(\mu)\phi_2(\xi,\mu)E(\xi)$ and integrate over $\mu\epsilon(0,1)$. Subtracting between the two equations we obtain

$$A_2(\xi) = A_2^0(\xi) + \xi N_2^{-1}(\xi)H_2^{-1}(\xi)\,X(\xi) \quad , \tag{72}$$

where

$$A^{0}(\xi) = \xi N_2^{-1}(\xi)\,\frac{c_2}{2}\,\frac{\nu_2}{\nu_2+\xi}\,H_2^{-1}(\xi)H_2^{-1}(\nu_2)\quad . \tag{73a}$$

and

$$\begin{aligned} X(\xi) = {} & A_1(\nu_1)\,\frac{c_1}{2}\,\frac{\nu_1}{\nu_1-\xi}\,H_2^{-1}(\nu_1)\,\{E(\nu_1)-E(\xi)\} \\ & + A_1(-\nu_1)\,\frac{c_1}{2}\,\frac{\nu_1}{\nu_1+\xi}\,H_2^{-1}(\nu_1)\,\{1-E(\nu_1)E(\xi)\} \\ & + \int_0^1 A_1(\nu)\,\frac{c_2-c_1}{2}\,\frac{\nu}{\xi-\nu}\,H_2(\nu)\,\{E(\nu)-E(\xi)\}\,d\nu \\ & + \int_0^1 A_1(-\nu)\,\frac{c_1}{2}\,\frac{\nu}{\nu+\xi}\,H_2^{-1}(\nu)\,\{1-E(\nu)E(\xi)\}\,d\nu \quad . \end{aligned} \tag{73b}$$

Equations (69), (70), and (72) can be solved numerically by a standard iterative method using the limiting value

$$\lim_{\nu\to\xi}\frac{\nu}{\xi-\nu}\{E(\nu)-E(\xi)\} = \frac{\alpha}{\xi}E(\xi) \tag{74}$$

in the third term in Eq.(73b).

We report in Figure 9 the scalar flux and in Table 8 the extrapolated endpoint for medium 2 given by Eq.(65). We note that if $c_1 > c_2$ the discrete coefficient for medium 2 tends to be positive, as shown in Table IX, and the usual concept of the extrapolated endpoint loses its meaning.

Table VIII

Problem 4, The Extrapolated Endpoint

c_1	c_2	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 5$
0.95	0.9	2.1173	–	–	–
0.8	0.9	1.4359	1.6881	1.7597	1.7821
0.6	0.9	1.1116	1.1656	1.1739	1.1754
0.9	0.8	4.0097	–	–	–
0.85	0.8	2.2966	–	–	–
0.6	0.8	1.3084	1.3957	1.4100	1.4126
0.6	0.7	1.6076	1.7825	1.8152	1.8214
0.5	0.7	1.4039	1.4755	1.4858	1.4875
0.5	0.6	1.7484	1.8964	1.9208	1.9249

Table IX

Problem 4, The Discrete Coefficient for Medium 2

c_1	c_2	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 5$
0.95	0.9	−0.10807	0.03671	0.10254	0.14707
0.9	0.8	−0.00336	0.09262	0.12598	0.14172
0.85	0.8	−0.03827	0.03142	0.05127	0.05853
0.8	0.7	0.00812	0.05641	0.06828	0.07189
0.7	0.6	0.00961	0.03484	0.03980	0.04095

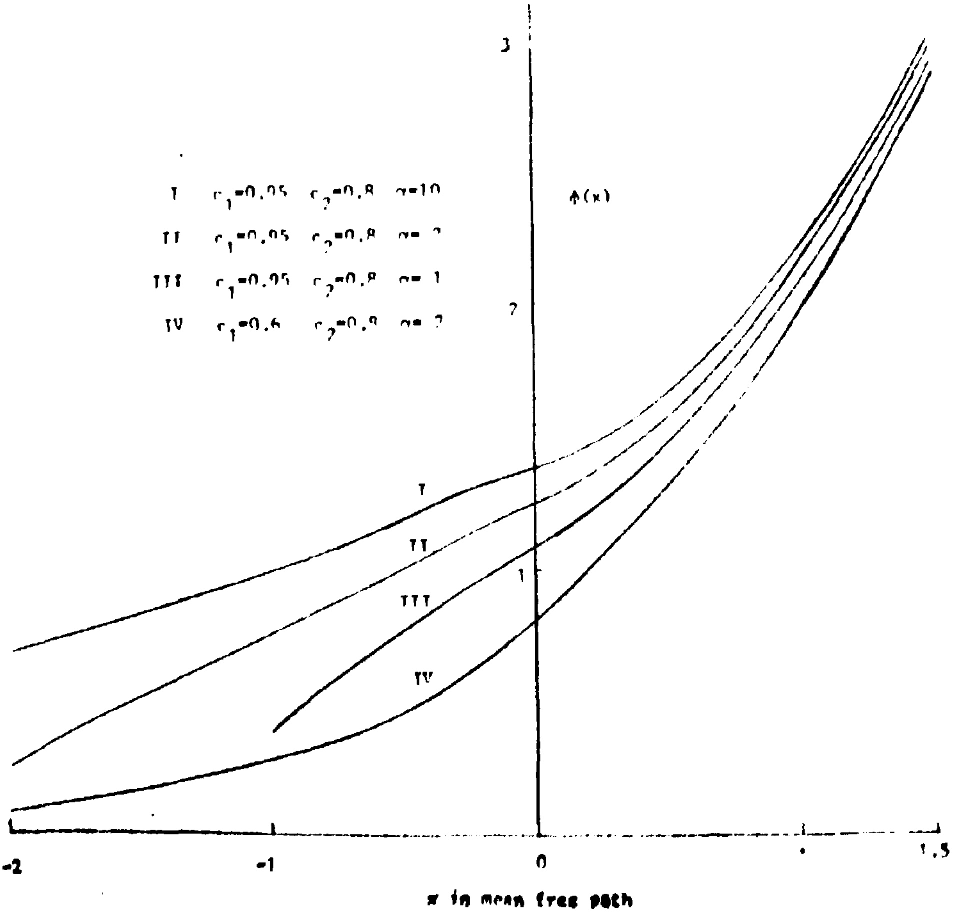

Figure 10 – Problem 4, The Scalar Flux

11 – TWO SLABS WITH AN INCIDENT FLUX: PROBLEM 5

Probably the most typical problem in two-slab geometry is the cell problem, i.e., the disadvantage factor calculation. Bond and Siewert(2) were able to remove singularities in the cell problem and reported numerical results. We consider here a different problem, two slabs irradiated by a flux of neutrons on from one side, as in Problem 2. The thickness of medium 1 $(0 \leqslant x \leqslant \alpha_1)$ is α_1, that of medium 2 $(\alpha_1 \leqslant x \leqslant \beta,\ \beta = \alpha_1 + \alpha_2)$ is α_2, and the incident flux is written as $f(\mu)$, $\mu \epsilon (0,1)$. This problem is more difficult that the cell problem due to the nonsymmetry. Bond and Siewert's technique cannot be applied to this problem and in fact no numerical solutions have been reported.

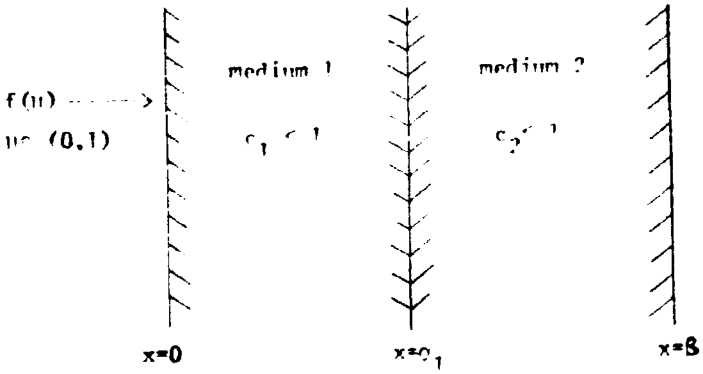

Figure 11 – Problem

The solutions can be written as

$$\psi_1(x,\mu) = A_1(\nu_1)\,\phi_1(\nu_1,\mu)\exp(-x/\nu_1) + A_1(-\nu_1)\,\phi_1(-\nu_1,\mu)\exp\{-(\alpha_1 - x)/\nu_1\}$$

$$+ \int_0^1 A_1(\nu)\,\phi_1(\nu,\mu)\exp(-x/\nu)\,d\nu + \int_0^1 A_1(-\nu)\,\phi_1(-\nu,\mu)\exp\{-(\alpha_1 - x)/\nu\}\,d\nu\,,$$

$$0 \leqslant x \leqslant \alpha_1\,, \tag{75a}$$

and

$$\psi_2(x,\mu) = A_2(\nu_2)\,\phi_2(\nu_2,\mu)\exp\{-(x-\alpha_1)/\nu_2\} + A_2(-\nu_2)\,\phi_2(-\nu_2,\mu)\exp\{-(\beta - x)/\nu_2\}$$

$$+ \int_0^1 A_2(\nu)\,\phi_2(\nu,\mu)\exp\{-(x-\alpha_1)/\nu\}\,d\nu + \int_0^1 A_2(-\nu)\,\phi_2(-\nu,\mu)\exp\{-(\beta - x)/\nu\}\,d\nu\,,$$

$$\alpha_1 \leqslant x \leqslant \beta \tag{75b}$$

The boundary condition at $x = 0$ and $x = \beta$ result in the equations

$$A_1(\nu_1)\,\phi_1(\nu_1,\mu) + \int_0^1 A_1(\nu)\,\phi_1(\nu,\mu)\,d\mu = f(\mu) - A_1(-\nu_1)\,\phi_1(-\nu_1,\mu)\,E_1(\nu_1)$$

$$\int_0^1 A_1(-\nu)\,\phi_1(-\nu,\mu)\,E_1(\nu)\,d\nu \quad , \quad \mu \in (0,1) \quad , \tag{76}$$

and from Eq.(23)

$$A_2(-\nu_2)\,\phi_2(\nu_2,\mu) + \int_0^1 A_2(-\nu)\,\phi_2(\nu,\mu)\,d\nu = -A_2(\nu_2)\,\phi_2(-\nu_2,\mu)\,E_2(\nu_2)$$

$$- \int_0^1 A_2(\nu)\,\phi_2(-\nu,\mu)\,E_2(\nu)\,d\nu \quad , \quad \mu \in (0,1) \quad , \tag{77}$$

where $E_i(\xi) = \exp(-\alpha_i/\xi)$, and we write the interface condition in two equations

$$A_1(-\nu_1)\,\phi_1(\nu_1,\mu) + \int_0^1 A_1(-\nu)\,\phi_1(\nu,\mu)\,d\nu = -A_1(\nu_1)\,\phi_1(-\nu_1,\mu)\,E_1(\nu_1)$$

$$- \int_0^1 A_1(\nu)\,\phi_1(-\nu,\mu)\,E_1(\nu)\,d\nu + A_2(\nu_2)\,\phi_2(-\nu_2,\mu)$$

$$+ A_2(-\nu_2)\,\phi_2(\nu_2,\mu)\,E_2(\nu_2) + \int_0^1 A_2(\nu)\,\phi_2(-\nu,\mu)\,d\nu$$

$$+ \int_0^1 A_2(-\nu)\,\phi_2(\nu,\mu)\,E_2(\nu)\,d\nu \quad , \quad \mu \in (0,1) \quad . \tag{78a}$$

and

$$A_2(\nu_2)\,\phi_2(\nu_2,\mu) + \int_0^1 A_2(\nu)\,\phi_2(\nu,\mu)\,d\nu = A_1(\nu_1)\,\phi_1(\nu_1,\mu)\,E_1(\nu_1)$$

$$+ A_1(-\nu_1)\,\phi_1(-\nu_1,\mu) + \int_0^1 A_1(\nu)\,\phi_1(\nu,\mu)\,E_1(\nu_1)\,d\nu$$

$$+ \int_0^1 A_1(-\nu)\,\phi_1(-\nu,\mu)\,d\nu - A_2(-\nu_2)\,\phi_2(-\nu_2,\mu)\,E_2(\nu_2)$$

$$- \int_0^1 A_2(-\nu)\,\phi_2(-\nu,\mu)\,E_2(\nu)\,d\nu \quad , \quad \mu \in (0,1) \quad . \tag{78b}$$

If we apply the half-range orthogonality relations to Eqs.(76) and (77), we can isolate the coefficients in the left side expansions to obtain

$$A_1(\xi) = A_1^0(\xi) - \xi N_1^{-1}(\xi) H_1^{-1}(\xi) Y_1(\xi) \quad , \quad \xi = \nu_1 \text{ or } \nu \in (0,1) \quad , \tag{79}$$

and

$$A_2(-\xi) = -\xi N_2^{-1}(\xi) H_2^{-1}(\xi) Y_2(\xi) \quad , \quad \xi = \nu_2 \text{ or } \nu \in (0,1) \quad , \tag{80}$$

where

$$A_1^{0}(\xi) = N_1^{-1}(\xi) H_1^{-1}(\xi) \int_0^1 \mu H_1(\mu)\, \phi_1(\xi,\mu)\, I(\mu)\, d\mu \quad , \tag{81a}$$

$$Y_1(\xi) = A_1(-\nu_1) \frac{c_1}{2} \frac{\nu_1}{\nu_1+\xi} H_1^{-1}(\nu_1)\, E_1(\nu_1) + \int_0^1 A_1(-\nu) \frac{c_1}{2} \frac{\nu}{\nu+\xi} H_1^{-1}(\nu)\, E_1(\nu)\, d\nu \quad , \tag{81b}$$

and

$$Y_2(\xi) = A_2(\nu_2) \frac{c_2}{2} \frac{\nu_2}{\nu_2+\xi} H_2^{-1}(\nu_2)\, E_2(\nu_2) + \int_0^1 A_2(\nu) \frac{c_2}{2} \frac{\nu}{\nu+\xi} H_2^{-1}(\nu)\, E_2(\nu)\, d\nu \quad . \tag{81c}$$

We next apply the half-range orthogonality relations for medium 1 to Eq.(78a) to isolate coefficients in the left side. The expression for the discrete coefficient $A_1(-\nu_1)$ is regular but the equation for the continuum coefficient $A_1(-\nu)$ includes a singular integral of $A_2(-\nu)$. To remove this singularity we multiply Eq.(77) by $\mu H_1(\mu)\phi_1(\xi,\mu)E_2(\xi)$ and integrate over $\mu \epsilon (0,1)$ and subtract this result from the previous equation to obtain

$$A_1(-\xi) = -\xi N_1^{-1}(\xi) H_1^{-1}(\xi) Y_3(\xi) \quad , \quad \xi = \nu_1 \text{ or } \nu \in (0,1) \quad . \tag{82}$$

where

$$\begin{aligned}
Y_3(\xi) = {} & A_1(\nu_1) \frac{c_1}{2} \frac{\nu_1}{\nu_1+\xi} H_1^{-1}(\nu_1)\, E_1(\nu_1) + \int_0^1 A_1(\nu) \frac{c_1}{2} \frac{\nu}{\nu+\xi} H_1^{-1}(\nu)\, E_1(\nu)\, d\nu \\
& - A_2(\nu_2) \frac{c_2}{2} \frac{\nu_2}{\nu_2+\xi} H_1^{-1}(\nu_2) \{ 1 - E_2(\nu_2)\, E_2(\xi) \} \\
& - A_2(-\nu_2) \frac{c_2}{2} \frac{\nu_2}{\nu_2-\xi} H_1^{-1}(-\nu_2) \{ E_2(\nu_2) - E_2(\xi) \} \\
& - \int_0^1 A_2(\nu) \frac{c_2}{2} \frac{\nu}{\nu+\xi} H_1^{-1}(\nu) \{ 1 - E_2(\nu)\, E_2(\xi) \}\, d\nu \\
& - \int_0^1 A_2(-\nu) \frac{c_1 - c_2}{2} \frac{\nu}{\nu-\xi} H_1(\nu) \{ E_2(\xi) - E_2(\nu) \}\, d\nu \quad .
\end{aligned} \tag{83}$$

In the same way we first apply the orthogonality relations for medium 2 to Eq.(78b), next multiply Eq.(76) by $\mu H_2(\mu)\,\phi_2(\xi,\mu)E_1(\xi)$ and integrate over $\mu\epsilon(0,1)$, and subtract between the two results to obtain

$$A_2(\xi) = A_1^0(\xi) - \xi N_2^{-1}(\xi)H_2^{-1}(\xi)\,(\xi) \quad , \quad \xi=\nu_2 \text{ or } \nu\epsilon(0,1) \tag{84}$$

where

$$A_2^0(\xi) = N_2^{-1}(\xi)H_2^{-1}(\xi)\,E_1(\xi)\int_0^1 \mu H_2(\mu)\,\phi_2(\xi,\mu)\,f(\mu)\,d\mu \quad . \tag{85a}$$

and

$$\begin{aligned} Y_4(\xi) = {} & A_1(\nu_1)\frac{c_1}{2}\frac{\nu_1}{\nu_1-\xi}H_2^{-1}(\nu_1)\{E_1(\nu_1)-E_1(\xi)\} \\ & - A_1(-\nu_1)\frac{c_1}{2}\frac{\nu_1}{\nu_1+\xi}H_2^{-1}(\nu_1)\{1-E_1(\nu_1)E_1(\xi)\} \\ & - \int_0^1 A_1(-\nu)\frac{c_1}{2}\frac{\nu}{\nu+\xi}H_2^{-1}(\nu)\{1-E_1(\nu)E_1(\xi)\}\,d\nu \\ & - \int_0^1 A_1(\nu)\frac{c_2-c_1}{2}\frac{\nu}{\nu-\xi}H_2(\nu)\{E_1(\xi)-E_1(\nu)\}\,d\nu \\ & + A_2(-\nu_2)\frac{c_2}{2}\frac{\nu_2}{\nu_2+\xi}H_2^{-1}(\nu_2)E_2(\nu_2) + \int_0^1 A_2(-\nu)\frac{c_2}{2}\frac{\nu}{\nu+\xi}H_2^{-1}(\nu)E_2(\nu)\,d\nu \quad . \end{aligned} \tag{85b}$$

Equations (79), (80), and (84) are the final equations for the coefficients to be solved by numerical iterations. We not that if $\alpha_2 \to 0$ or if $c_2 = 0$, Eqs.(79) and (82) reduce to Eqs.(49), the case of single slab.

We consider two cases of incident flux

$$f(\mu) = \mu \qquad \text{Case 1} \tag{86a}$$

and

$$F(\mu) = \mu^2 \quad , \qquad \text{Case 2} \; . \tag{86b}$$

The integrals in Eqs.(81a) and (85a) can be expressed in terms of moments of the H functions in a form similar to Eqs.(54).

The transmission rate defined as

$$\delta = \int_0^1 \psi_2(\beta,\mu)\,\mu d\mu \;/\; \int_0^1 f(\mu)\,\mu d\mu \tag{87}$$

is reported in Table X and the total flux is shown in Figure 12 for several sets of parameters.

Table X

Problem 5, The transmission Rate

c_1	c_2	α_1	α_2	δ (Case 1)	δ (Case 2)
0.95	0.8	1	1	0.26754	0.27988
0.8	0.95	1	1	0.27194	0.28688
0.8	0.95	1	2	0.17634	0.18658
0.8	0.95	2	1	0.13383	0.14246
0.8	0.7	1	1	0.19359	0.20582
0.8	0.7	1	2	0.08355	0.08962
0.8	0.7	2	1	0.09439	0.10086
0.7	0.8	1	1	0.19554	0.20893
0.7	0.6	1	1	0.15822	0.17006
0.6	0.7	1	1	0.15971	0.17244
0.6	0.7	1	2	0.06922	0.07549
0.6	0.7	2	1	0.06303	0.06898
0.6	0.5	1	1	0.13379	0.14527
0.6	0.5	1	2	0.04908	0.05407
0.6	0.5	2	1	0.05286	0.05805
0.5	0.6	1	1	0.13497	0.14716

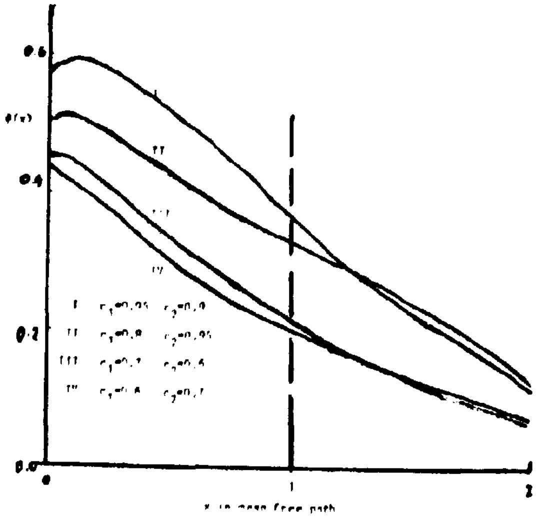

Figure 12 – Problem 5, The Scalar flux in two slabs with an incident flux $\psi(0,\mu) = \mu^2, \mu \in (0,1)$

12 – THE CASE OF LINEARLY ANISOTROPIC SCATTERING

Although the analyses for the case of anisotropic scattering are more involved and the algebra can be quite tedious, the development of the theory is at about the same stage, at least for the case of linearly anisotropic scattering, as for isotropic scattering. More recent works in Refs. 3 and 10-13.

Here we summarize basic formulas and report some numerical results.

The general solution of the transport equation

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \psi(x,\mu) = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu')\,(1 + b\mu\mu')\,d\mu' \qquad (88)$$

can be given as

$$\psi(x,\mu) = A(\nu_0)\,\phi(\nu_0,\mu)\exp(-x/\nu_0) + A(-\nu_0)\,\phi(-\nu_0,\mu)\exp(x/\nu_0)$$

$$+ \int_0^1 A(\nu)\,\phi(\nu,\mu)\exp(-x/\nu)\,d\nu + \int_0^1 A(-\nu)\,\phi(-\nu,\mu)\exp(x/\nu)\,d\nu \ , \qquad (89)$$

where A's are expansion coefficients, $\pm\nu_0$ are discrete eigenvalues,

$$\phi(\pm\nu_0,\mu) = \frac{c\nu_0}{2}\, R(\pm\nu_0\,\mu)\,\frac{1}{\nu_0 \mp \mu} \qquad (90a)$$

and

$$\phi(\nu,\mu) = \frac{c\nu}{2}\, R(\nu\mu)\,\frac{P}{\nu-\mu} + \lambda(\nu)\,\delta(\nu-\mu) \quad , \quad \nu \in (-1,1) \ , \qquad (90b)$$

with

$$R(x) = 1 + rx \quad , \quad r = b(1-c) \ , \qquad (91a)$$

and

$$\lambda(\nu) = 1 - c\nu R(\nu^2)\tanh^{-1}\nu + cr\nu^2 \ . \qquad (91b)$$

The full-range and half-range completeness theorems have been established and the half-range orthogonality theorem can be summarized, in terms of the Chandrasekhar H function, as

$$\int_0^1 \tilde{\phi}(\nu_0, \mu)\, \phi(\nu, \mu)\, \mu H(\mu)\, d\mu = 0 \tag{92a}$$

$$\int_0^1 \text{[illegible]}\, \mu H(\mu)\, d\mu = 0 \quad , \tag{92b}$$

$$\int_0^1 \tilde{\phi}(\nu_0, \mu)\, \phi(\nu_0, \mu)\, \mu H(\mu)\, d\mu = H(\nu_0)\, N(\nu_0) \tag{92c}$$

and

$$\int_0^1 \tilde{\phi}(\nu, \mu)\, \phi(\nu', \mu)\, \mu H(\mu)\, d\mu = H(\nu)\, N(\nu)\, \delta(\nu - \nu') \quad , \tag{92d}$$

where $\nu, \nu' \in (0,1)$, $\nu_0 > 1$ or $\nu_0 = i\,|\nu_0|$, and

$$\tilde{\phi}(\xi, \mu) = \phi(\xi, \mu) + \frac{c\xi}{2}\, d \quad , \tag{93}$$

where

$$d = c\nu H_1 (2 - cH_0)^{-1} \quad , \quad H_i = \int_0^1 H(\mu) \mu^i\, d\mu \quad . \tag{94a,b}$$

The functions $N(\nu_0)$ and $N(\nu)$ are known explicity[3] and the H function satisfies the equations

$$\frac{cz}{2} \int_0^1 H(\mu)\, R(\mu^2)\, \frac{1}{\mu + z}\, d\mu = 1 - H^{-1}(z) \quad , \quad z \notin (-1,0) \quad , \tag{95a}$$

$$\frac{c\nu}{2} \int_0^1 H(\mu)\, R(\mu^2)\, \frac{P}{\nu - \mu}\, d\mu = 1 - H(\nu)\, \lambda(\nu) \quad , \quad \nu \in (0,1) \quad , \tag{95b}$$

and

$$\frac{c\nu_0}{2} \int_0^1 H(\mu)\, R(\mu^2)\, \frac{1}{\nu_0 - \mu}\, d\mu = 1 \tag{95c}$$

Integrals involving eigenfunctions with negative eigenvalues can be summarized in the following formula:

$$\int_0^1 \tilde{\phi}(\xi,\mu)\,\phi(\xi',\mu)\,\mu H(\mu)\,d\mu = \frac{c}{2}\,\frac{\xi\xi'}{\xi+\xi'}\,H^{-1}(\xi')\,\{1 + r\xi\xi' + d(\xi+\xi')\}\ ,$$

$$\xi,\xi' = \nu_c \text{ or } \nu \in (0,1) \tag{96}$$

involving eigenfunctions for different media are as follows

$$\int_0^1 \tilde{\phi}_i(\xi,\mu)\,\phi_j(\xi',\mu)\,\mu H_i(\mu)\,d\mu = \frac{c_i}{2}\,\frac{\xi\xi'}{\xi+\xi'}\,J_1(i,\xi;j,\xi') \tag{97a}$$

$$\int_0^1 \tilde{\phi}_i(\xi,\mu)\,\phi_j(\nu_j,\mu)\,\mu H_i(\mu)\,d\mu = \frac{c_i}{2}\,\frac{\xi\nu_j}{\nu_i-\xi}\,J_1(i,\xi;j,\nu_j)\ , \tag{97b}$$

$$\int_0^1 \tilde{\phi}_i(\nu_i,\mu)\,\phi_j(\nu,\mu)\,\mu H_i(\mu)\,d\mu = \frac{1}{2}\,\frac{\nu_i\nu}{\nu_j-\nu}\,J_2(i,\nu_i;j,\nu)\ , \tag{97c}$$

$$\int_0^1 \tilde{\phi}_i(\nu,\mu)\,\mu H_i(\mu)\int_0^1 \phi_j(\nu',\mu)\,A(\nu')\,d\nu'\,d\mu$$

$$= \int_0^1 A(\nu')\,\frac{\nu}{2}\,\frac{\nu' P}{\nu-\nu'}\,J_2(i,\nu;j,\nu')\,d\nu'$$

$$+\ A(\nu)\,H_i(\nu)\,\nu\,\{\lambda_i(\nu)\,\lambda_j(\nu) + \frac{1}{2}\,c_i\,c_j\,\pi^2\,\nu^2\,R_j(\nu^2)\,R_j(\nu^2)\}\ , \tag{97d}$$

where subscripts i and j refer to the media, ν_i and ν_j are discrete eigenvalues,

$$J_1(i,x;j,y) = \{1 + r_i xy + d_i(x-y)\}\,R_i^{-1}(y^2)\,R_j(y^2)\,H_i^{-1}(-y)$$

$$-\ 2(1-c_j)\,(2-c_i H_{io})^{-1}\,y(x-y)\,R_i^{-1}(y^2)\,(r_i - r_j)\ , \tag{98a}$$

and

$$J_2(i,x;j,y) = \{c_i - c_j + y^2\,[c_i r_i - c_j r_j - c_i c_j\,(r_i - r_j)]\}\,\{1 + r_i xy + d_i(x-y)\}\,R_i^{-1}(y^2)\,H_i(y)$$

$$+\ 2c_j(1-c_i)\,(2-c_i H_{io})^{-1}\,y(x-y)\,R_i^{-1}(y^2)\,(r_i - r_j) \tag{98b}$$

To report some numerical examples we consider two slabs with an incident flux, the same problem as in Section 11. The analysis is just the same as for isotropic scattering except, of course, that relevant formulas must be used and that the final equations are more involved. We report in Table XI the transmission rate for $\alpha_1 = \alpha_2 = 1$ for three cases of incident flux

$$f(\mu) = 2 \qquad \text{Case 1} \tag{99a}$$

$$f(\mu) = 3\mu \qquad \text{Case 2} \; , \tag{99b}$$

and

$$f(\mu) = 4\mu^2 \qquad \text{Case 3} \; , \tag{99c}$$

and in Table XII the total flux for Case 3.

Table XI

The Case of Anisotropic Scattering

The Total Flux for Case 3 with $c_1 = 0.9$, $c_2 = 0.8$, $\alpha_1 = \alpha_2 = 1$

x	$b_1 = 0.0$ $b_2 = 0.0$	$b_2 = 0.0$ $b_2 = 0.5$	$b_1 = 0.5$ $b_2 = 0.0$	$b_1 = 0.5$ $b_2 = 0.5$
0.0	2.163458	2.146073	2.112900	2.093217
0.1	2.227643	2.205926	2.192401	2.168329
0.2	2.179463	2.154132	2.160544	2.132906
0.4	2.004640	1.972364	2.016685	1.982277
0.6	1.792831	1.753478	1.833099	1.791854
0.8	1.569029	1.522204	1.635046	1.586629
1.0	1.332774	1.278094	1.422135	1.366233
1.2	1.111723	1.080908	1.184632	1.153705
1.4	0.923606	0.911571	0.983206	0.971915
1.6	0.752907	0.756258	0.800882	0.805652
1.8	0.591327	0.607549	0.628622	0.646798
2.0	0.413115	0.439349	0.438940	0.467462

13 – CONCLUDING COMMENTS

We first note that our final equations for two media problems reduce to the case of a single medium if the tickness of either medium is reduced to zero or in the limit $c \to 0$ and that in our method all coefficients are found at the same time, which is not the case with the method of Ref. 14. In this sense and in that only the half-range orthogonality relations are required, the method used is a natural extension of the one that has been established for single-medium (half-space and slab) problems.

Tabela XII

Case of Anisotropic Scattering
The Transmission Rate for $\sigma_1 = \sigma_2 = 1$

c_1	b_1	c_2	b_2	δ (Case 1)	δ (Case 2)	δ (Case 3)
0.9	0.0	0.8	0.0	0.227958	0.248241	0.260885
0.9	0.0	0.8	0.5	0.241515	0.262939	0.276277
0.9	0.5	0.8	0.0	0.242383	0.263576	0.276732
0.9	0.5	0.8	0.5	0.257168	0.279586	0.293484
0.8	0.0	0.6	0.0	0.156877	0.174740	0.186307
0.8	0.0	0.6	0.5	0.165172	0.183895	0.195999
0.8	0.5	0.6	0.0	0.167151	0.185701	0.197650
0.8	0.5	0.6	0.5	0.176138	0.195600	0.208114
0.6	0.0	0.5	0.0	0.116412	0.133790	0.145274
0.6	0.0	0.5	0.5	0.121953	0.140078	0.152037
0.6	0.5	0.5	0.0	0.122898	0.140776	0.152535
0.6	0.5	0.5	0.5	0.128812	0.147469	0.159721

One disadvantage of the method is that it does not solve two-half-space problems analytically in one-group theory, as can be done by the use of the two-media orthogonality relations or by the method of Ref. 14. One may find another in that some of the regularized equations become quite long, involving all unknown coefficients, as the number of regions increases. What is ultimately desired, excluding analytical solutions of multiregion problems, is a sort of two-media orthogonality relations that regularize all coefficients at an interface resulting in a set of equations in each of which coefficients for only two adjacent media appear. The two-media orthogonality relations available at present are short of this goal and, considering the fact that even the orthogonality relation of this type have not been found in two-group theory, the prospect for attaining this goal in various models appears slight.

In conclusion we note that we are concerned here not with analytical solutions but only with the problem of reducing the set of singular integral equations to one of regular integrals, considering the present situation that single-slab problems in one-group theory have not been solved analytically; that our method can be used for this purpose in various problems, though in some problems there are simpler ways to achieve it; that numerical solutions of the resulting equations seem to pose no difficulty; and that our method requires only half-range orthogonality relations which can be derived by the invariant-imbedding technique.

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