Transactions of the **13th International Conference on Structural Mechanics in Reactor Technology** (SMiRT 13), Escola de Engenharia - Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil, August 13-18, 1995

The solution of ill-posed inverse problems in solid mechanics and the nuclear industry

Bezerra, L.M.
CNEN/SP IPEN - Comissão Nacional de Energia Nuclear, São Paulo, SP, Brazil

ABSTRACT: In mathematical physics a well-defined problem is generally governed by a differential equation (ordinary or partial), valid in a certain domain, and additional conditions called boundary conditions. In solid mechanics a well-defined problem is characterized by (1) the domain of the body and its boundary, (2) the governing equilibrium equation, (3) the appropriate boundary conditions, (4) the material properties involved in the governing equation, and (5) forces and displacements acting on the body. When any of the items before is missing, incomplete, or overdefined, the problem is regarded as an inverse problem also called ill-posed inverse problem (IP). This paper instead of presenting particular mathematical formulations to solve specific ill-posed inverse problem shows the state of the art in the literature, the difficults, applications, and the importance of computational techniques for the solution of such problems in solid mechanics. It also describes the potentials that such numerical techniques have in applications in the nuclear industry. In brief, this paper tries to motivate the engineering community for more research in the area of inverse problems.

1 INTRODUCTION

The analysis of well-posed problems has progressed for more than two centuries resulting in a superb knowledge of techniques for exact and numerical solutions of such problems. The same is not true for IP. Only in recent years ill-posed inverse problems have received attention from the engineering community (Weber 1981). In problems of solid mechanics, given a well defined domain and sufficient boundary condition, the displacement, strain, and stress fields at any interior point of the body may be obtained from the solution of the boundary value problem. Such problems with a well-defined set of boundary conditions, are termed as direct problems. The existence and uniqueness of the solution for such direct problems has been studied by Sokolnikoff (1956) and Muskhelishvili (1963), among others. In practice, however, there are situations where the domain and/or boundary conditions may be incomplete, and one wishes to determine them based on experimental measurements. This type of problem is classified as ill-posed inverse problem.

Much of the literature on the solution of ill-posed problems has been devoted to the area of heat conduction. However, other applications of inverse problems in recent years have been noticed in geophysics, seismology, image processing (Tikhonov and Gonchanrsky 1987), and biomedical engineering (Pilkington 1982, Rudy et al. 1992), among others. In solid mechanics, a large class of ill-posed inverse problems fall under two categories, namely the reconstruction problems and the identification problems (Baumeister 1981). Reconstruction problems refer to the determination of missing boundary conditions or some other input to the system with known geometry. Such input results in an output which is previously known from experimental data. Identification problems deal with the determination of the unknown geometry of the system, e.g.; internal defects, for which the boundary conditions are prescribed and for which the measured responses at selected points are given.

2 RESEARCHES IN IP

The beginning of the space era has played an important role in the development of numerical solutions for inverse problems in engineering. The limitation on the placement of sensors on the surfaces of reentry vehicles to measure temperature was crucial (Beck et al. 1985, Hensel and Hill 1989). There are situations where it is impossible to place temperature sensors directly on the surface of spacecrafts because of severe aerodynamic heating. It is then necessary to embed the sensors beneath the surface of the aircrafts and infer the boundary condition (temperature, heat flux) from interior data (Hensel 1991). The problem of estimating surface heat flux from interior temperature measurements is known as the Inverse Heat Conduction Problem (IHCP) in the literature (Beck et al. 1985). Varied approaches have been proposed in the literature for the solution of IHCPs. These have included the use of Duhamels's theorem (Stolz 1960, Beck 1968, 1979, Hills and Mulholland 1979) (or convolution integral), and the use of Laplace transform (Woo and Chow 1981). However, such techniques have limited use for realistic problems (Beck et al., 1985). Numerical procedures, such as finite differences (Beck 1970, Blackwell 1981, Willians and Curry 1977), finite elements (Hensel and Hills 1989, Tandy et al. 1986, Busby and Trujillo 1985), and integral methods (Zabaras et al. 1989a, Dorri 1987, Ohnaka and Uosaki 1985) have also been employed for the solution of ill-posed inverse problems due to their ability to treat a large variety of complex problems. Before studying inverse problems in solid mechanics, it is wise to know the efforts that have been done in the area of IHCP. An extensive survey can be found in the books by Beck (1985) and Hensel (1991), among others.

The solutions of inverse problems in solid mechanics have not been widely attempted so far. Among the published works, more attention has been given to the reconstruction inverse problems than to the identification inverse problems. The following description illustrates the works done in both areas.

Sutton (1988) calculated the boundary displacements using least-square minimization schemes in conjunction with the boundary element method. This procedure was unable to treat problems with even a small percentage error in the data unless an extremely large number of data points was used. Grysa (1981) applied the Laplace transform method to problems in thermoelasticity to solve for boundary temperatures from internal displacements or stresses for a one-dimensional problem.

Gao and Mura (1989a, 1989b, 1991) have presented an integral equation formulation for the non-destructive evaluation of interfacial quantities in composites or around plastic regions, They have applied such formulation to simple cases using zero order regularization techniques. Kubo (1988) reviewed the recent progress of inverse problems related to the mechanics and fracture of solids.

Tanaka and Masuda (1986) presented a boundary integral formulation for flaw detection in Laplace's potential formulation. Tanaka, Nakamura and Nakano (1988) also treated the problem of flaw detection in elastodynamics with multiple force excitations using the boundary element in conjunction with the conjugate gradient method for optimization but it was concluded that no convergence was obtained when more than two design variables were treated simultaneously. In such approaches a finite difference approximation was used to obtain the response sensitivities with respect to the design variables. The deficiencies of such an approximation in obtaining reliable solutions have been pointed out by Saigal, Aithal and Kane (1989).

Maniatty et al. (1989) performed some studies on the inverse elastic problems. They have employed the finite element method and a first order spatial regularization scheme using the measurements of internal strains as well as the displacements to solve for the boundary traction reconstruction in terms of shape and magnitude. Schnur and Zabaras (1990) presented the boundary condition reconstruction, and the 'keynode method', which consists of specifying a polynomial to represent the missing boundary conditions. Zabaras, Moreles, and Schnur (1989b) also used the BEM for the same problem of boundary condition reconstruction. In all their papers no optimization methods were employed.

Bezerra and Saigal (1991, 1993, 1994) presented a boundary element approach with implicit differentiation of the sensitivities in conjunction with the BFGS (Reklaitis 1983) optimization algorithm and obtained very good results for flaw detection in planar structures. Such formulation treated the constraint equations augmenting them as internal penalty functions to the objective function to be minimized. Using a similar formulation, Bezerra and Saigal (1993, 1995) also solved the problem of boundary traction reconstruction detecting not only the magnitude but also the extent and position of the missing boundary tractions.

Schnur and Zabaras (1992), using the FEM, presented a paper on the detection of elastic inclusions in objects using penalty functions and a modified Levenberg-Marquardt optimization method to match the measured displacements to a finite element model solution which depends on the unknown parameters. Trujillo (1985) proposed dynamic programming as a general method for the solution of inverse problems. Using the FEM and the dynamic programming (an alternative optimization approach), the formulation was applied to IHCP and to structural frame dynamics for two-dimensional cases with good results. However, for problems with higher dimensionality, dynamic programming is not appropriate (Bellman 1957).

3 EXAMPLES & MOTIVATIONS OF IP

Recently, there has been an increased interest in the application of the inverse algorithms to the solution of problems in engineering and science. This is due to lower computer costs, higher speed and memory capacity, and the existence of more accurate experimental measurement techniques (Tikhonov and Goncharsky 1987). There is a great potential in combining computer technology, via numerical modeling, and experimental methods. The motivation behind that is to make more effective use of their respective strengths and to overcome their separate limitations. Experimental methods of holography and speckle interferometry, among others, yields measurements of surface displacements of a structure. On the other hand, the determination of stress, based on experimental measuremts, requires differentiation of the displacements and can result in a substantial loss of accuracy. Also regions of maximum stress are sometimes regions of relatively little displacement; thus experimental data may be sparse in those areas. However, numerical analyses (e.g. FEM or BEM) are often capable of accurate stress evaluation but requires an accurate knowledge of boundary conditions. Relatively little research has been devoted to developing an integrated approach between experimeltal and numerical techniques for stress analysis (Weathers et al. 1985).

Let's examine a few examples of ill-posed problems and the respective motivations to solve them.The problem of detecting defects such as flaws. It is a topic of great interest in fracture mechanics (Kubo 1988) and in composite materials for aerospace structures (Banks and Kojima 1988). In experimental mechanics, various techniques are available for NDT of objects (Hull and John 1988), such as: visual inspection, penetrant liquid, magnetic particles, eddy current, ultrasonic, and x-rays.

Visual inspection and penetrant liquid can only detect surface-breaking defects, while magnetic particles can detect surface-breaking defects and give indications of subsurface defects in ferro-magnetic materials only. Eddy current techniques are applicable to conducting materials only, and while they can sense surface and subsurface flaws, they are not able to identify them. Ultrasonic testing may sometimes detect the position and relative size of the flaws (Figure 1a) but is not able to predict the shape. The DGS diagram may be used to estimate a disc-shaped reflector but cannot be used to size defects unless ideally oriented to the ultrasonic beam (Halmshaw 1987). Many precautions must also be taken for the successful application of the ultrasonic techniques, for example, the surface where the probes are applied usually have to be smoothed in order to avoid scattering effects. Attempts to assign a real size from ultrasonic techniques is still a subject of many research papers (Halmshaw 1987). In many cases, such as in the detection of weld defects, doubtful situations may arise and radiography tests are usually required to confirm the suspicious defect. In austenitic steel the problems are even more difficult mainly due to anisotropic grain structure, which causes the ultrasonic beam to bend.

Only x-ray techniques are capable of effectively detecting the size, shape and location of defects (Broek 1986). However, radiographic techniques possess numerous drawbacks. They tend to be expensive and the capital costs for fixed x-ray equipment are high and considerable space is needed for radiography laboratory, including a dark room for film processing. For portable x-ray sets, the capital costs are less but space for film processing and interpretation is still required. Radiographic inspection of components on sites may be a tardy process because the portable x-ray equipment is usually limited to a relatively low energy radiation emission, and many shots are necessary to inspect an object. In addition the x-ray costs may be increased by the need to protect personnel from the effects of radiation. The safety aspects will apply not only to the personnel directly concerned in the test, but also to all people who work in the vicinity of a radiography test. Moreover, defects laying in a direction parallel to x-ray beam may escape detection (Figure 1b). Also, the inaccessibility of the region to be investigated may also prohibit the use of x-ray. It is felt that computational techniques for the detection of flaws may be cheaper than x-ray and more adequate than DGS planar estimations in ultrasonic testing.

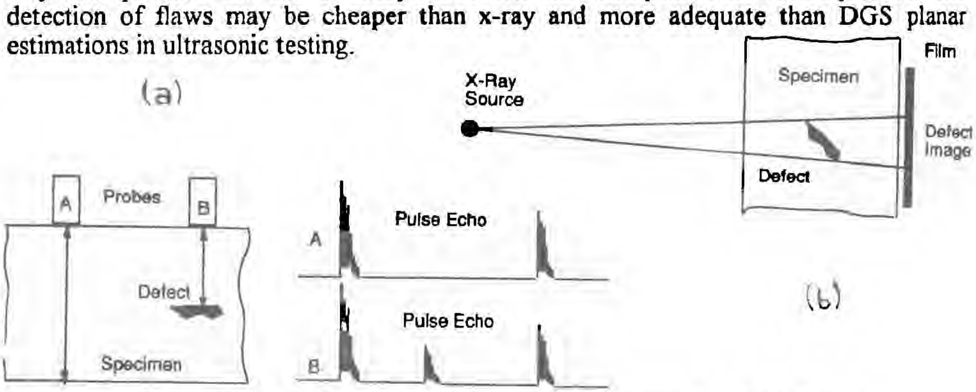


Figure 1 - Detection of defects using: (a) Ultrasound and (b) X-ray

Other area of recent interest, that also deserves our attention, concerns the ill-posed inverse problems of reconstructing missing boundary condition, in particular, a missing boundary traction (Maniatty et al. 1989, Zabaras et al. 1989b, 1990). The state of deformation, strain, and stress within a body may be determined only when a sufficient set of boundary tractions is defined. Sometimes, the complete set of boundary tractions may not be available and it may be necessary to reconstruct the boundary tractions using response measurements from sensors located at discrete, interior or exterior locations of a solid.The inverse problem of boundary condition reconstruction in solid mechanics may be employed in characterizing tractions at inaccessible regions of critical components in sensitive mechanical equipment, or characterizing tractions on a portion of the body embedded in a hazardous environment like inside NPP. Another application is the determination of tractions in a physical truncation. Such tractions may also be needed in a partial mesh discretization of the body (Barishpolsky 1981). It is noted that in such cases, the internal data are generally not only more accurate but also easier to assess. Techniques like strain gages, photoelasticity, coating, and speckle interferometry, among others, are reliable experimental methods for the determination of strain, deformation, and stress tensor at the boundary of or inside a body (Weathers et al. 1985, Balas et al. 1983, Dally and Riley 1965). Computational techniques for the solution of the inverse problem of boundary traction reconstruction may provide an evaluation tool, such as identifying contact regions in neighboring objects, as well as hybrid experimental and numerical methods for the analyses of solids (Weathers et al. 1985, Balas et al. 1983).

Among other areas of study of ill-posed inverse problem in solid mechanics, one can include the problem of finding material properties and material interface of inclusions inside a material matrix. The solution of such problem has practical application in non-destructive evaluation and detection of material inclusion properties and strees interfaces. Generally, numerical solution for such problem represents the calculation of the position and radius of a circular inclusion in a finite plane matrix as well as the young's moduli of the inclusion and the matrix. To make the solution of such problem easier, it is generally assumed that both matrix and inclusion are homogeneous, elastic and isotropic. Numerical methods for the determination of material properties and interfaces between regions of damage in solids and structures integrate optimization and finite element analysis and also automatic mesh generation and update. Another problem in the area of dynamics of structure is the problem of identifying input time histories from displacement and velocities measurements taken from sensors located at discrete points of a structure. Also the problem of finding the equivalent material properties - for simple structural systems, like beam elements - given a set of natural frequencies has deserved attention from the engineering community.

4 SOME SCHEMES FOR SOLVING IP

The principal schemes tried for IP solutions are (Kubo 1988): (a) measurement values are substituted into the governing equations. Simultaneous equations for the unknown parameters are constructed. (b) direct analysis can be made using ordinary schemes assuming values for the parameters to be identified. A combination of parameter values that gives the best fit of the calculated response to the data vector is employed as the inverse solution. (c) Laplace or Fourier transforms of the parameters is calculated from the measured physical quantities. The unknown parameters are calculated using the corresponding inverse transform.

In the first scheme a system of matrix equations is obtained, but unstable solutions due to small errors in the data vector may generate ill-conditioned system of matrix equations. Inverse problems are extremely sensitive to errors in the data vector. There are a number of procedures for the solution of such problems in general. One of these procedures was developed by Tikhonov (1977). He introduced the regularization method to reduce the sensitivity of ill-posed problems to measurement errors present at the sensors (Beck, 1985). The third scheme may be applied only to small problems where analytical solutions can be found.

The second scheme is preferred by numerous investigators since it has more numerical stability when used in conjunction with robust optimization methods. The basis of this scheme is to minimize a residual function that measures the differences between the model prediction and the reality in terms of experimental data (Figure 2). A number of metrics can be employed to express the residual between experimental and predicted values. In general, the difference between the model value mapping $\psi = AZ$ and the experimental data vector $\hat{\psi}$, to be minimized, can be written as

$$f(Z) = \left[|AZ - \hat{\psi}|^q \right]^{\frac{1}{q}}; \quad q \geq 1 \qquad (1)$$

Ill-posed inverse problems are more difficult to solve than well-posed problems. The existence, uniqueness and stability of the solution of ill-posed problems are not guaranteed. The minimization of a residual function that measures the differences between the model prediction and the experimental data involves finding the values of the unknowns in a mathematical model such that the response calculated with the model matches the measured data.

Recently, algorithms have been developed that combine optimization methods to match the data with numerical methods such as the finite element method or the boundary element method to solve the model. To overcome the ill-posedness of the inverse problem, a priori information on the required solution is often introduced in the form of smoothness conditions (with regularization or approximating functions) or as quantitative constraints on the solution. The main advantage of this approach is that it can be applied to a wide variety of problems in science and engineering.

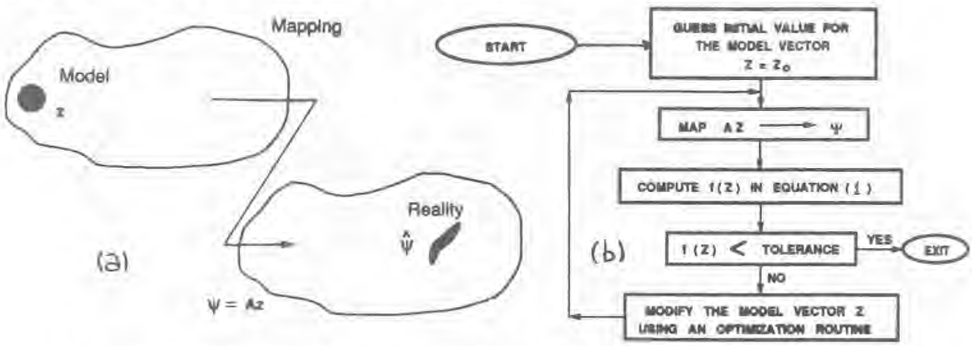


Figure 2 - (a) Mapping between model and reality. (b) A flow chart for IP.

No general method exists in the engineering literature for the solution of IP since these problems are of a wide variety. The finite element method (FEM), and the boundary element method (BEM) which have been successfully applied to direct problems, are gradually being incorporated into schemes for solving IP.

In problems with no body force BI formulations reduces by one the dimensionality of the problem and involves only the meshing of the boundaries of the body (Banerjee and Butterfield 1981). For IPs which continuously require an updating of the mesh, a "meshless" like the boundary element nethod is better when compared with other methods that involve discretizing the entire continuum. Less discretization leads to a system of equations with smaller number of d.o.f.. This is particularly advantageous for ill-posed problems because few d.o.f. implies more numerical stability in the solution process (Dori 1987). A shortcoming of the BEM is the mathematical complexity underlying the method which is not observed in FEM formulations. In addition, due to the descretization all over the body the FEM promptly can give the state of strain and stress in the entire continuum while in the BEM the users, a priori, have to specify the internal points of interest. FEM also has some distinctive advantages over BEM for non linear problems (Zienkiewicz 1977).

5 ROBUSTNESS AND ERRORS

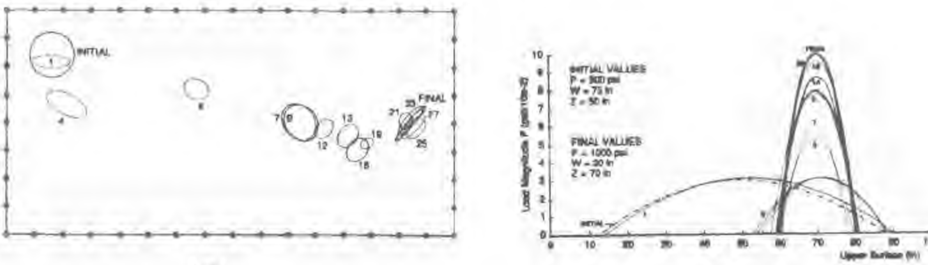
An algorithm for the solution of ill-posed problems in solid mechanics should predict accurate solutions if the measured data are of high accuracy. The insertion of small errors in the measurement data $\hat{\psi}_{ik}$ may cause instability in the numerical solution of IP (Tikhonov and Goncharsky 1987). A robust algorithm should be stable for small errors contaminating the measured quantities in the data vector $\hat{\psi}_{ik}$ (Beck et al. 1985). Errors may be separated into systematic and random components. Common systematic errors are those constant errors caused by such effects as sensitivity shift, zero offset, calibration, and known nonlinearities. Repeating the measurements under different conditions, or with different equipments, often eliminate the systematic errors. Generally, for simplification in the numerical procedures to solve IP, any known effects (Cerni and Foster 1962) due to equipment, installation, interference, and systematic calibration errors are assumed to be removed to the extent that remaining errors in $\hat{\psi}_{ik}$ may be considered random. Therefore, displacements, stresses, or strains in the measurement data are assumed to primarily contain random errors. In addition, it is assumed that these random errors can be statistically described. It is noted that the quantities $\hat{\psi}_{ik}$ are measured at discrete locations since only a finite number of sensors are available. Moreover, the errors distribution in the data can have a significant influence on the solution of ill-posed problems.

In general, a set of standard assumptions to consider for the solution of IP include (Beck et al. 1985, Bezerra 1993): (a) Errors are additive, and therefore a random quantity can be added to "true" values of the experimental data. (b) The mean value of the random errors is zero. This implies that there is no bias in the data, and that the errors are both positive and negative such that the mean value of the random errors turns out to be zero. (c) A constant variance is assumed. This implies that the random error variance is independent of the position where the measurements are taken. (d) Zero correlation is assumed among the errors. Therefore the measurement errors at different points have no correlation between each other. The error in one sensor does not affect the data from another sensor. (e) Errors have a normal or Gaussian distribution. It is important to notice that classical least-squares inversion tends to be rather sensitive to departure from normalcy. A very interesting article on robust methods in inverse theory was presented by Scales and Gersztenkon (1988)

6 INVERSE SOLUTIONS & THE NUCLEAR INDUSTRY

Many areas could take advantage of the computational techniques for the solution of ill-posed inverse problems in solid mechanics. In the nuclear industry, for example, due to the hazardous nuclear radioactivity it is dangerous to go inside certain areas of a NPP to make examination, for example, for the detection of flaws in critical nuclear equipments.

The placement of sensors at discrete points of pipings and equipments could provide the necessary on line data to feed computer interfaces. With such data, one could monitor crack initiation and growth, for example. The on line data could be displacement, stress, or strain measurements obtained when the piping or the equipment is under known deterministic loading or multiple force excitations. It is obvious that the present stage of research & development of IP solution algorithms is not sufficient to provide something like that. However some interesting results are available in the literature. That's why more research in the area of ill-posed inverse problems are very welcome. Figure 3, for example, shows a typical detection of a flat elliptical flaw in a panel under bending loading. Starting from an initial guess the flaw model evolved to the final expected configuration. Such result was obtained with a boundary element approach with implicit differentiation of the sensitivities in conjunction with an optimization algorithm. With more investment in the area of defect detection, it will be possible, in the future, to find defects in structural systems with methods similar to the compuratized tomography approach.



(a) (b)

Figure 3 - (a) Panel with bending loading. Evolution from the circular guess to the final elliptical flaw (b) Parabolic traction distribution reconstruction (determination of magnitude, span and location) (Bezerra & Saigal 1991, 1993).

Sometimes it is also necessary to know the bearing stress in regions of difficult access. In such a case, given some response of the structure under prescribed loading condition and some measurements of displacements, stresses, or strains in the vicinity of the region of interest, it is possible to find the bearing stress. The determination of such contact stresses is possible using computational techniques for IP in conjunction with experimental measurements. Such techniques can also be useful for the determination of the friction coefficient between two touching surfaces. Figure 3b shows an example of contact stress reconstruction. In that example not only the magnitude but also the span and location of the contact stress were determined.

To illustrate two more applications of computational algorithms for the solution of ill-posed inverse problems associated with experimental techniques, see Figures 4 and 5. Figure 4b shows the fringe patterns of the principal-stress difference near a circular hole obtained by FEA (Figure 4a) using only discrete vertical and horizontal displacements given at the boundary of the body. The true fringe distribution is shown in Figure 4c. The displacements used in the FEA were obtained using laser-speckle interferometry. Such results were taken from the interesting paper written by Weathers et al. (1985). This example clearly shows the possibility of integration between numerical schemes (FEA) with on line data from sensors and experimental techniques. With such approach, one could monitor the stress distribution in a critical nuclear equipment. Another example of IP solution can be seen in Figure 5. That figure was taken from the good paper written by Trujillo (1978) proposing a general method for IP solution. In that figure, a frame with an unknown force f(t) acting on it is rebuilt from "measured" velocities taken at nodes z3 and z5. Figure 5b shows the true force exiting the frame and in Figure 5c the inverse solution of such force. The results obtained are very good. Other examples are available in the literature but due to space limitation it is impossible to show all of them

7 CONCLUSIONS

Instead of presenting mathematical formulations, the author had preferred to describe the principal problems encountered in IP, the state of the art in the area, and to show the importance of computational techniques for the solution of ill-posed inverse problems in solid mechanics. Such computational techniques may provide an effective tool for non-destructive testing, examination, monitoring, and diagnosis as well as hybrid experimental and numerical techniques for the analysis of nuclear equipments or structural systems in the nuclear industry. In particular, due to the hazardous consequences of radioactive doses on human being, numerical approaches for the solution of the ill-posed inverse problems in solid mechanics could become an important mean to protect man from hazardous environment during the regular inspections of nuclear equipments. Finally, it should be noticed that ill-posed inverse problems, despite it numerous physical applications, have received scant attention from the engineering community. The reason for that is that ill-posed inverse problems, from the mathematical point of view, are not "well behaved" problems. The existence, uniqueness, and stability of the solutions for ill-posed inverse problems are not at all guaranteed.

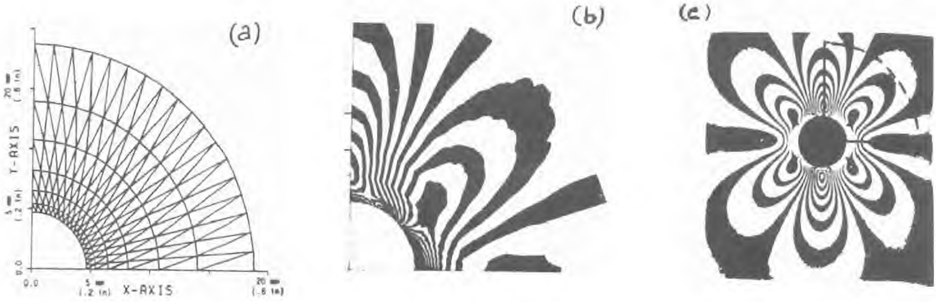


Figure 4 - Stress distribution around a hole (Weathers et al. 1985).

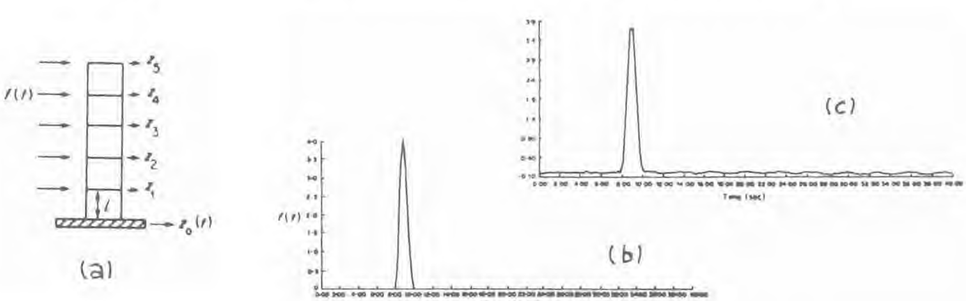


Figure 5 - Reconstruction of the forcing function (Trujjilo 1978).

REFERENCES

Balas, J., J. Sladek, and M. Drzik. 1983. Stress analysis by combination of holographic interferometry and boundary integral method. Exper. Mech., 23: 196-202.

Banerjee, P. K., and R. Butterfield. 1981. Boundary Element Methods in Engineering Science. London: McGraw-Hill.
Banks, H. T., and F. Kojima. 1988. Boundary shape identification problems in two dimensional domains related to thermal testing of materials. ICASE Report No. 88-23. NASA Langley Research Center.
Baumeister J. 1981. Stable solution of inverse problem. Vieweg, Braun-schweig.
BarishpolskyB. M. 1980. A combined experimental and numerical method for the solution of generalized elasticity problem. Exper. Mech. 20: 345-349. 1981
Beck J. V., B. Blackwell, and C. R. ST. Clair Jr.. 1985. Inverse Heat Conduction Ill-posed Problems. New York: Wiley-Interscience Publication.
Bellman R.. 1957. Dynamic Programming. Princeton: Princeton University Press.
Blackwell B. F.. 1981. Efficient techniques for the numerical solution of the one dimensional inverse problem of heat transfer. Num. Heat Transfer. 4: 229-239
Bezerra L. M., and S. Saigal. 1991. Flaw detection in elastostatics with boundary element. 22nd Midwestern Mechanics Conference. Oct, 6-9, U. of Missouri, Rola.
Bezerra L. M., and S. Saigal. 1993a. A boundary element formulation for the inverse elastostatics problem (IESP) of flaw detection. Int. J. Numer. Methods in Eng.
Bezerra L. M., and S. Saigal 1993b. Integral equation formulation for boundary data reconstruction in elastostatics. Structural Optimization 93 - World Congress, Aug. 2-6, Rio, Brazil.
Bezerra L. M.. 1993c. Inverse Elastostatics Solutions with Boundary Elements. Ph.D. Thesis. Cranegie Mellon University. Pittsburgh, PA, USA.
Bezerra L M., and S. Saigal. 1994. A boundary Integral Formulation for the inverse problem of detecting open cracks. Pressure Vessel and Piping Conference, June 19-23, Minneapolis, Minnesota, USA.
Bezerra L M., and S. Saigal. 1995. Inverse boundary traction reconstruction with the BEM. Int. Journal of Solids and Structures (in press).
Broek D. 1986. Elementary Engineering Fracture Mechanics. 4th ed. Dordrecht: Martinus Nijhoff Publishers.
Cerni R. H., and L. E. Foster, 1962. Instrumentation For Emgineering Measurement. New York: John Wiley.
Dally J. W., and W. F. Riley. 1965. Experimental Stress Analysis. NY: McGraw-Hill.
Dori B. 1987. Solution of inverse heat conduction problems using boundary integral methods. General Eletric CR&D Report.
Gao Z. and T. Mura. 1989a. On the inversion of residual stress from surfaces displacements. J. Appl. Mech.. 56: 508-513.
Gao Z. and T. Mura. 1989b. Nondestructive evaluation of interfacial damages in composite materials. Int. J. Solids Structures. 25: 901-916.
Gao Z. and T. Mura. 1991. Elasticity problems with partially overspecified boundary conditions. Int. J. Engng. Sci.. 29: 685-692.
Grysa K., M. J. Cialkowski, and H. Kaminski. 1981. An inverse temperature field problem of the theory of thermal stresses. Nuc. Eng. Des.. 64: 169-184.
Halmshaw R. 1987. Non-Destructive Testing. Vitoria, Australia: Edward Arnold.
Hensel E., and R. Hills. 1989. Steady-state two-dimensional inverse heat conduction. Numerical Heat Transfer. 15: 227-240.
Hensel E.. 1991. Inverse Theory and Applications for Engineers. Englewood Cliffs, New Jersey: Prentice-Hall.
Hills R. J., and G. P. Mulholland. 1979. Accuracy and resolving power of one-dimensional transient inverse heat conduction theory as applied to discrete and inacurate measurements. Int. J. Heat Mass Transfer. 22: 1221-1229.
Hull J. B. and V. B. John. 1988. Non-Destructive Testing. New York: Springer-Verlag.
Kubo S. 1988. Inverse problems related to mechanics and fracture of solids and structures. JSME Int. Jornal. 31: 157-166.

Maniatty A., N. Zabaras, and K. Stelson. 1989. Finite element analysis of some inverse elasticity problems. J. of Eng. Mech.. 115: 1303-1317.
Muskhelishvili, 1963. Some Basic Problems of the Mathematical Theory of Elasticity. The Netherlands: P. Noordhoff Ltd.
Ohnaka K., and K. Uosaki. 1985. Boundary element approach for identification of boundary conditions of distributed-parameter system. Int. J. Control 41: 981-990.
Pilkington, T. C. 1982. Engineering Contribution to Biophysical Electrocardiography. New York: Spring-Verlag.
Reklaitis G. V., A. Ravindran, and K. M. Ragsdell. 1983. Engineering Optimization - Methods and Applications. New York: John Wiley.
Rudy Y., and H. S. Oster. 1992. The electrocardiographic inverse problem. Critical Rev. in Biomedical Engng. 28: 2795-2911.
Saigal S., R. Aithal, and J. H. Kane. 1989. Conforming boundary elements in plane elasticity for shape design sensitivities. Int. Journal for Numerical Methods in Engineering. 28: 2795-2911.
Scales J. A., and A. Gerstenkorn. 1988. Robust methods in inverse problems. Inverse Problems. 4: 1071-1091.
Schnur D. S., and N. Zabaras. 1990. Finite element solution f two-dimensional inverse elastic problems using spatial smoothing. Int. Journal for Numerical Methods in Engineering 30: 57-75.
Schnur D. S., and N. Zabaras. 1992. An inverse method for determining elastic material properties and material interface. Int. J. for Num. Methods in Engineering 30: 57-75.
Stolz G. Jr. 1960. Numerical solution to an inverse problem of heat conduction for simple shapes. J. Heat Transfer. 82: 20-26.
Sutton, M. A. et al. 1988. A study of the use of measured displacement data with integral equations of elasticity (submited to Engrg. Anal. J.).
Sokolnikoff, I. S. 1956. Mathematical Theory of Elasticity. New York: McGraw-Hill.
Trujillo D. M.. 1978. Application of dynamic programming to the general inverse problem. Int. Journal for Numerical Methods in Engineering. 12: 613-624.,
Tanaka M., and Y. Masuda. 1986. Boundary element method applied to some inverse problems. Engineering Analysis. 3: 138-143.
Tanaka M., M. Nakamura, and T. Nakano. 1988. Defect shape identification by means of elastodynamics boundary element analysis and optimization techniques. Advances in Boundary Elements. Ed. C. A. Brebbia. Berlin: Spring-verlag. 3: 183-194.
Tikhonov A. N., and V. Y. Arsenin. 1977. Solutions of Ill-Posed Problems. New York: John Wiley.
Tikhonov and Goncharsky. 1987. Ill-Posed Problems in the Natural Sciences. Moscow: MIR Publishers
Weathers J. M., W. A. Foster, W. F. Swinson, and J. L. Turner. 1985. Integration of laser-speckle and finite element techniques of stress analysis. Exper. Mech. 25: 60-65.
Weber C. F. 1981. Analysis and slution of the ill-posed inverse heat conduction problem. Int. Heat Mass Transfer. 24: 1783-1792.
Willians S. D., and D. M. Curry. 1977. An analytical experimental study for surface heat flux determination. J. Spacecraft. 14: 632-637.
Woo K. C., and L. C. Chow. 1981. Inverse heat conduction by direct inverse Laplace Transform. Num. Heat Transfer. 4: 499-504.
Zabaras N., S. Mukherjee, and O. Richmond. 1989a. An analysis of inverse heat transfer problems with phase change using integral method. Trans. of ASME. 110: 554-560.
Zabaras N., Morellas V., and Schnur D. 1989b. Spatially regularized solution of inverse elasticity problems using the BEM. Comm. in Appl. Num. Methods. 5: 547-553.
Zienkiewicz O. C.. 1977. The Finite Element Method. 3rd edition. McGraw-Hill.

Maniatty A., N. Zabaras, and K. Stelson. 1989. Finite element analysis of some inverse elasticity problems. J. of Eng. Mech.. 115: 1303-1317.
Muskhelishvili, 1963. Some Basic Problems of the Mathematical Theory of Elasticity. The Netherlands: P. Noordhoff Ltd.
Ohnaka K., and K. Uosaki. 1985. Boundary element approach for identification of boundary conditions of distributed-parameter system. Int. J. Control 41: 981-990.
Pilkington, T. C. 1982. Engineering Contribution to Biophysical Electrocardiography. New York: Spring-Verlag.
Reklaitis G. V., A. Ravindran, and K. M. Ragsdell. 1983. Engineering Optimization - Methods and Applications. New York: John Wiley.
Rudy Y., and H. S. Oster. 1992. The electrocardiographic inverse problem. Critical Rev. in Biomedical Engng. 28: 2795-2911.
Saigal S., R. Aithal, and J. H. Kane. 1989. Conforming boundary elements in plane elasticity for shape design sensitivities. Int. Journal for Numerical Methods in Engineering. 28: 2795-2911.
Scales J. A., and A. Gerstenkorn. 1988. Robust methods in inverse problems. Inverse Problems. 4: 1071-1091.
Schnur D. S., and N. Zabaras. 1990. Finite element solution f two-dimensional inverse elastic problems using spatial smoothing. Int. Journal for Numerical Methods in Engineering 30: 57-75.
Schnur D. S., and N. Zabaras. 1992. An inverse method for determining elastic material properties and material interface. Int. J. for Num. Methods in Engineering 30: 57-75.
Stolz G. Jr. 1960. Numerical solution to an inverse problem of heat conduction for simple shapes. J. Heat Transfer. 82: 20-26.
Sutton, M. A. et al. 1988. A study of the use of measured displacement data with integral equations of elasticity (submited to Engrg. Anal. J.).
Sokolnikoff, I. S. 1956. Mathematical Theory of Elasticity. New York: McGraw-Hill.
Trujillo D. M.. 1978. Application of dynamic programming to the general inverse problem. Int. Journal for Numerical Methods in Engineering. 12: 613-624.,
Tanaka M., and Y. Masuda. 1986. Boundary element method applied to some inverse problems. Engineering Analysis. 3: 138-143.
Tanaka M., M. Nakamura, and T. Nakano. 1988. Defect shape identification by means of elastodynamics boundary element analysis and optimization techniques. Advances in Boundary Elements. Ed. C. A. Brebbia. Berlin: Spring-verlag. 3: 183-194.
Tikhonov A. N., and V. Y. Arsenin. 1977. Solutions of Ill-Posed Problems. New York: John Wiley.
Tikhonov and Goncharsky. 1987. Ill-Posed Problems in the Natural Sciences. Moscow: MIR Publishers
Weathers J. M., W. A. Foster, W. F. Swinson, and J. L. Turner. 1985. Integration of laser-speckle and finite element techniques of stress analysis. Exper. Mech. 25: 60-65.
Weber C. F. 1981. Analysis and slution of the ill-posed inverse heat conduction problem. Int. Heat Mass Transfer. 24: 1783-1792.
Willians S. D., and D. M. Curry. 1977. An analytical experimental study for surface heat flux determination. J. Spacecraft. 14: 632-637.
Woo K. C., and L. C. Chow. 1981. Inverse heat conduction by direct inverse Laplace Transform. Num. Heat Transfer. 4: 499-504.
Zabaras N., S. Mukherjee, and O. Richmond. 1989a. An analysis of inverse heat transfer problems with phase change using integral method. Trans. of ASME. 110: 554-560.
Zabaras N., Morellas V., and Schnur D. 1989b. Spatially regularized solution of inverse elasticity problems using the BEM. Comm. in Appl. Num. Methods. 5: 547-553.
Zienkiewicz O. C.. 1977. The Finite Element Method. 3rd edition. McGraw-Hill.