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LINEARIZED BOLTZMANN EQUATION**

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C. E. Siewert*

ABSTRACT

The elementary solutions and the half-range completeness and orthogonality theorems concerning the linearized Boltzmann equation are discussed.

I - INTRODUCTION

We wish to consider here the linearized Boltzmann equation written, for steady state conditions, as⁽¹⁾

$$(c_1 \frac{\partial}{\partial x_1} + 1) h(x_1, c_1, c_2, c_3) = \int h(x_1, c'_1, c'_2, c'_3) K(\underline{c}' : \underline{c}) e^{-c'^2} d^3 c' \quad (1)$$

Here x_1 , c_1 , c_2 and c_3 are, respectively, the non dimensional space variable and velocity components, \underline{c} is the velocity of the particles and $c = |\underline{c}|$. The dependent variable h represents the perturbation of the particle distribution function from the Maxwellian⁽¹⁾. In addition, the scattering kernel is taken here to be

$$K(\underline{c}' : \underline{c}) = \frac{1}{\pi^{3/2}} \left[1 + 2\underline{c}' \cdot \underline{c} + \frac{2}{3} (c'^2 - \frac{3}{2}) (c^2 - \frac{3}{2}) \right]. \quad (2)$$

Since we are interested here in temperature-density effects, we can take "moments" of Eq. (1) to obtain equations dependent only on x_1 and c_1 . Thus we let

$$\Psi_1(x_1, c_1) = \pi^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(c_2^2 + c_3^2)} h(x_1, c_1, c_2, c_3) dc_2 dc_3 \quad (3a)$$

and

$$\Psi_2(x_1, c_1) = \pi^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(c_2^2 + c_3^2)} h(x_1, c_1, c_2, c_3) (c_2^2 + c_3^2 - 1) dc_2 dc_3 \quad (3b)$$

so that the density perturbation

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$$\Delta N(x_1) = \pi^{-3/2} \int h(x_1, c_1, c_2, c_3) e^{-c^2} d^3c \quad (4)$$

and the temperature perturbation

$$\Delta T(x_1) = \frac{2}{3} \pi^{-3/2} \int h(x_1, c_1, c_2, c_3) (c^2 - \frac{3}{2}) e^{-c^2} d^3c \quad (5)$$

can be expressed as

$$\Delta N(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Psi_1(x, \mu) e^{-\mu^2} d\mu \quad (6)$$

and

$$\Delta T(x) = \frac{2}{3\pi} \int_{-\infty}^{\infty} [(\mu^2 - \frac{1}{2})\Psi_1(x, \mu) + \Psi_2(x, \mu)] e^{-\mu^2} d\mu, \quad (7)$$

where we have used x for x_1 and μ for c_1 . If now we integrate Eq. (1) from $-\infty$ to ∞ over both c_2 and c_3 and then multiply Eq. (1) by $(c_2^2 + c_3^2 - 1)$ and integrate similarly, we find that the resulting two coupled equations can be written as

$$\mu \frac{\partial}{\partial x} \underline{\Psi}(x, \mu) + \underline{\Psi}(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} [\underline{Q}(\mu) \underline{\tilde{Q}}(\mu') + 2\mu\mu' \underline{P}] \underline{\Psi}(x, \mu') e^{-\mu'^2} d\mu', \quad (8)$$

where $\underline{\Psi}(x, \mu)$ is a two-vector with elements $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$,

$$\underline{Q}(\mu) = \begin{vmatrix} (\frac{2}{3})^{1/2} (\mu^2 - \frac{1}{2}) & 1 \\ (\frac{2}{3})^{1/2} & 0 \end{vmatrix} \quad \text{and} \quad \underline{P} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}. \quad (9)$$

We note that we can deduce from Eq. (8) that $\underline{J}_1(x)$, where

$$\underline{J}_1(x) = \int_{-\infty}^{\infty} \mu e^{-\mu^2} \underline{\Psi}(x, \mu) d\mu, \quad (10)$$

is a constant and thus this term can effectively be removed⁽²⁾ from the equation to give the equation studied by Kriese, Chang and Siewert⁽³⁾ in a paper here-after referred to as KCS.

II – HALF-SPACE PROBLEMS

We note that the elementary solutions of Eq. (8) and the required half-range completeness and orthogonality theorems concerning the elementary solutions have been reported in KCS. Here we wish simply to review those results expressed in a slightly improved form.

As reported by KCS, a general solution of Eq. (8) can be written as

$$\Psi(x, \mu) = \sum_{\alpha=1}^2 A_{\alpha} F_{\alpha}(\mu) + \sum_{\alpha=3}^4 A_{\alpha} \Psi_{\alpha}(x, \mu) + \sum_{\alpha=1}^2 \int_{-\infty}^{\infty} A_{\alpha}(\eta) F_{\alpha}(\eta, \mu) e^{-x/\eta} d\eta, \quad (11)$$

where

$$\underline{F}_1(\mu) = \underline{Q}(\mu) \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \quad \underline{F}_2(\mu) = \underline{Q}(\mu) \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \quad (12a)$$

$$\underline{\Psi}_3(x, \mu) = (\mu - x) \underline{F}_1(\mu) \quad \text{and} \quad \underline{\Psi}_4(x, \mu) = (\mu - x) \underline{F}_2(\mu), \quad (12b)$$

and

$$\underline{F}_{\alpha}(\eta, \mu) = \frac{1}{\sqrt{\pi}} \left[\eta \left(\frac{P}{\eta - \mu} \right) + \lambda_{\alpha}^*(\eta) \delta(\eta - \mu) \right] \underline{Q}(\mu) \underline{M}_{\alpha}(\eta). \quad (13)$$

Here

$$[\underline{\lambda}(\eta) - \lambda^*(\eta) \underline{\Psi}(\eta)] \underline{M}(\eta) = \underline{0}, \quad (14)$$

$$\underline{\lambda}(\eta) = \underline{I} + \eta P \int_{-\infty}^{\infty} \underline{\Psi}(\mu) \frac{d\mu}{\mu - \eta}, \quad (15)$$

$$\underline{\Psi}(\eta) = \frac{1}{\sqrt{\pi}} \underline{\tilde{Q}}(\eta) \underline{Q}(\eta) e^{-\eta^2}, \quad (16)$$

and the dispersion matrix is

$$\underline{\Lambda}(z) = \underline{I} + z \int_{-\infty}^{\infty} \underline{\Psi}(\mu) \frac{d\mu}{\mu - z}. \quad (17)$$

If we use the continuum expansion coefficients to define a vector

$$\underline{A}(\eta) = A_1(\eta) \underline{A}_1(\eta) + A_2(\eta) \underline{A}_2(\eta) \quad (18)$$

and let

$$\underline{A}_+ = \sqrt{\pi} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad \text{and} \quad \underline{A}_- = \sqrt{\pi} \begin{bmatrix} A_3 \\ A_4 \end{bmatrix}, \quad (19)$$

then we can express the general solution of Eq. (8) as

$$\underline{\Psi}(x, \mu) = \underline{\Phi}(\mu) \underline{A}_+ + (\mu - \lambda) \underline{\Phi}(\mu) \underline{A}_- + \int_{-\infty}^{\infty} \underline{\Phi}(\eta, \lambda) \underline{A}_-(\eta) e^{-x/\eta} d\eta, \quad (20)$$

where

$$\underline{\Phi}(\mu) = \frac{1}{\sqrt{\pi}} \underline{Q}(\mu) \quad (21)$$

and

$$\underline{\Phi}(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta \left(\frac{P}{\eta - \mu} \right) \underline{Q}(\mu) + \delta(\eta - \mu) e^{\eta^2} \tilde{Q}^{-1}(\eta) \underline{A}_-(\eta). \quad (22)$$

In KCS a half-range expansion theorem was proved, and thus we can state here that the equation

$$\underline{I}(\mu) = \underline{\Phi}(\mu) \underline{A}_+ + \int_0^{\infty} \underline{\Phi}(\eta, \mu) \underline{A}_-(\eta) d\eta, \quad \mu \in (0, \infty), \quad (23)$$

has a solution for all Hölder continuous functions $\underline{I}(\mu)$. Also in KCS a half-range orthogonality theorem was deduced; this allows us to write

$$\int_0^{\infty} \tilde{\theta}(\eta', \mu) \underline{\Phi}(\mu) e^{-\mu^2} \mu d\mu = \underline{0}, \quad \eta' > 0, \quad (24a)$$

$$\int_0^{\infty} \tilde{\theta}(\eta', \mu) \tilde{\phi}(\eta, \mu) e^{-\mu^2} \mu d\mu = \underline{N}(\eta) \delta(\eta - \eta') \quad , \quad \eta', \eta > 0, \quad (24b)$$

$$\int_0^{\infty} \tilde{\theta}(\mu) \tilde{\phi}(\eta, \mu) e^{-\mu^2} \mu d\mu = \underline{0} \quad , \quad \eta > 0, \quad (24c)$$

and

$$\int_0^{\infty} \tilde{\theta}(\mu) \tilde{\phi}(\mu) e^{-\mu^2} \mu d\mu = \underline{N}_+ \quad . \quad (24d)$$

Here the adjoint matrices are

$$\tilde{\theta}(\eta', \mu) = \tilde{\Phi}(\eta', \mu) \tilde{Q}^{-1}(\mu) \tilde{H}^{-1}(\eta') \tilde{H}(\mu) \tilde{Q}(\mu) \quad (25a)$$

and

$$\tilde{\theta}(\mu) = \pi^{-1/2} \tilde{H}(\mu) \tilde{Q}(\mu), \quad (25b)$$

where $\tilde{H}(\mu)$ is the unique solution⁽³⁾ of

$$\tilde{H}(\mu) = \underline{I} + \mu \tilde{H}(\mu) \int_0^{\infty} \tilde{H}(\eta) \tilde{\Psi}(\eta) \frac{d\eta}{\eta + \mu} \quad , \quad \mu \in [0, \infty), \quad (26a)$$

and

$$\int_0^{\infty} \tilde{H}(\mu) \tilde{\Psi}(\mu) d\mu = \underline{I} \quad . \quad (26b)$$

In addition, the normalization vectors are given by

$$\underline{N}(\eta) = \pi^{-1/2} \{ [\underline{\lambda}(\eta) \underline{\Psi}^{-1}(\eta) \underline{\lambda}(\eta) + \pi^2 \eta^2 \underline{\Psi}(\eta)] \} \quad (27a)$$

and

$$\underline{N}_+ = \pi^{-1/2} \int_0^{\infty} \tilde{H}(\mu) \tilde{\Psi}(\mu) \mu d\mu. \quad (27b)$$

To complete our review, we note that a typical half-space problem can be solved concisely in terms of the established formalism. For example, we can write a solution of Eq. (8) that is bounded at infinity and satisfies the free-surface condition

$$\underline{\Psi}(0, \mu) = \underline{\Psi}_{inc}(\mu) \quad , \quad \mu \in (0, \infty), \quad (28)$$

as

$$\underline{\Psi}(x, \mu) = \underline{\Phi}(\mu) \underline{A}_+ + \int_0^\infty \underline{\Phi}(\eta, \mu) \underline{A}(\eta) e^{-x/\eta} d\eta, \quad (29)$$

where

$$\underline{A}_+ = \underline{N}_+^{-1} \int_0^\infty \underline{\theta}(\mu) \underline{\Psi}_{inc}(\mu) e^{-\mu^2} \mu d\mu \quad (30a)$$

and

$$\underline{A}(\eta) = \underline{N}^{-1}(\eta) \int_0^\infty \underline{\theta}(\eta, \mu) \underline{\Psi}_{inc}(\mu) e^{-\mu^2} \mu d\mu. \quad (30b)$$

If we set $x = 0$ in Eq. (29) and consider only negative μ , then we can write

$$\underline{\Psi}(0, -\mu) = \underline{\Phi}(-\mu) \underline{A}_+ + \int_0^\infty \underline{\Phi}(\eta, -\mu) \underline{A}(\eta) d\eta \quad , \quad \mu > 0. \quad (31)$$

Upon substituting Eqs. (30) into Eq. (31), we find that the integration over η can be performed analytically to yield the concise surface result

$$\underline{\Psi}(0, -\mu) = \int_0^\infty \underline{R}(\mu' \rightarrow \mu) \underline{I}_{inc}(\mu') d\mu' \quad , \quad \mu > 0, \quad (32)$$

where

$$\underline{R}(\mu' \rightarrow \mu) = (\pi)^{-1/2} \frac{\mu'}{\mu' + \mu} \underline{Q}(\mu) \underline{H}(\mu) \underline{\tilde{H}}(\mu') \underline{\tilde{Q}}(\mu') e^{-\mu'^2}. \quad (33)$$

RESUMO

São discutidas as soluções elementares e teoremas de completude e ortogonalidade da Equação de Boltzmann Linearizada.

RÉSUMÉ

On a discuté les solutions élémentaires et théorèmes de complétion et d'orthogonalité de l'équation de Boltzmann linéarisée.

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