Higher Order Sampling and Recovering of Lowpass Signals

Gessé Eduardo Calvo Nogueira and Ademar Ferreira

***Abstract*—A higher order sampling function applied to lowpass signals is described. To recover the original signal from its periodic nonuniform samples, a formulation of the sampling function is introduced, based on the occurrence of local gaps in the sampled signal spectrum. As a result, the interpolant is greatly simplified.**

***Index Terms*—High order sampling, nonuniform interpolators, nonuniform sampling.**

I. INTRODUCTION

The Nth-order sampling of lowpass signals is used in several areas, such as in medical pulsed Doppler ultrasonic velocimeters [1] and in multirate systems [2]. In the literature, known methods[1] for signal reconstruction from its periodic nonuniform samples are often accomplished by N interpolators (see, e.g., [3]). The implementation of such interpolators, which are often approximated by FIR filters, requires various considerations. Vaughan *et al.* [4], for example, showed that the length and the dynamic range (number of bits) needed for digital implementation of the interpolators proposed by Kohlenberg [5] (second-order sampling) for the lowpass case, are progressively larger as the Kohlenberg's factor k departs from its optimum value: $k = T/2$ (uniform sampling), T is the sampling interval of each uniform sequence, where the interpolator becomes a lowpass filter. Although, to our knowledge, a comparative study on the performance of such interpolators, concerning filter length, computational efficiency, etc., is not found in the literature, one can verify that the known periodic nonuniform interpolation methods require considerably more complicated processing than the one for uniform sampling.

This correspondence describes in Section II a new higher order sampling function and restrictions to this sampling pattern intended to simplify the signal reconstruction. In Section III, restrictions are derived to recover second-order sampled lowpass signals at minimum average sampling rate by using only lowpass filters and a frequency shifter. Third and fourth-order sampling cases are also discussed in Section IV, where the signal reconstruction is accomplished using only lowpass filters but with average sampling rate larger than the minimum.

II. HIGHER ORDER SAMPLING

Given a real and lowpass signal $s(t)$ bandlimited to the frequency interval $|f| < B$ Hz, i.e., its highest spectral component is located at $B' < B$, we define the Nth-order sampling of this signal as

$$s_N(t) = s(t) \cdot \sum_{i=1}^{N} e_i \sum_{n=-\infty}^{\infty} \delta(t - nT - k_i) \tag{1}$$

Manuscript received February 15, 1996; revised January 12, 2000. This work was supported by the Programa de Apoio ao Desenvolvimento Científico e Tecnológico (PADCT) and by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). The associate editor coordinating the review of this paper and approving it for publication was Dr. Nurgun Erdol.

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Publisher Item Identifier S 1053-587X(00)04925-4.

[1]Some of these methods are described for bandpass signals, of which lowpass signals are a special case.

where $i = 1, 2, \cdots, N$, $n = \pm 1, \pm 2, \cdots$, e_i are amplitudes (nonzero real values), T is the sampling period of each uniform sampling sequence, each one presenting a delay k_i, and $\delta(t)$ is the Dirac delta function.

The average sampling frequency in (1) is $\overline{f}_S = N/T$. Here, we refer to $\overline{f}_S$ as minimum when it is equal to the Nyquist rate, i.e., when $\overline{f}_S = 2B$. To simplify the following analysis, we impose $k_1 = 0$ and $0 < k_i < T$ for $i \neq 1$. The following notation is used: $S(f)$ designates the Fourier transform of $s(t)$; since $s(t)$ is real and lowpass, its spectrum can be represented by $S(f) = S^+(f) + S^-(f)$, where $S^+(f)$ and $S^-(f)$ occupy, respectively, positive and negative frequencies, from here on referred to as upper and lower bands, correspondingly, of $S(f)$ or any of its replicas. The Fourier transform of (1) is

$$S_N(f) = \sum_{i=1}^{N} \frac{e_i}{T} \sum_{n=-\infty}^{\infty} S(f - n/T) \exp\left(-j2\pi n \frac{k_i}{T}\right). \tag{2}$$

The replicas of $S(f)$ in (2), in general, are $1/T$ spaced apart. For $N = 1$ (uniform sampling case), obviously, the recovery of $s(t)$ is possible if $B \leq 1/2T$. For $N > 1$, if the spacing between consecutive replicas, even locally, can be made larger than $1/T$, then the band of $s(t)$ can be increased. We discuss now some conditions to obtain this enlargement by generating gaps between consecutive replicas.

Notice in (2) that at every frequency n/T, the phases and magnitudes of the N replicas can be such that the resulting sum is constructive or destructive, depending on the amplitudes e_i and delays k_i. When destructive, that is, when

$$\sum_{i=1}^{N} \frac{e_i}{T} S\left(f - \frac{n}{T}\right) \exp\left(-j2\pi n \frac{k_i}{T}\right) = 0 \tag{3}$$

we say that a gap occurs in this location of the spectrum. In this case, the empty spectral spacing between the two remaining replicas adjacent to this gap, situated at $n-1$ and $n+1$, is magnified to $2/T$. For two consecutive gaps, the empty spacing is $3/T$, and so forth. Once one gap is produced, to recover the original signal, the selection of bands of the replicas adjacent to the generated gap is affected by spectral windowing, as will be shown in the next section.

III. RECOVERY OF SECOND-ORDER SAMPLED SIGNALS

To originate one gap at a specific frequency n/T, one can impose the condition (3) to this position. Instead of this, in the case $N = 2$, it is easy to search for all possible gaps at all positions n in (2), as we show next.

For second-order sampling, the spectrum of (2) is

$$S_2(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \left[e_1 + e_2 \exp\left(-j2\pi n \frac{k_2}{T}\right)\right] S\left(f - \frac{n}{T}\right). \tag{4}$$

To find e_1, e_2, and k_2 for all possible n positions in (4) that satisfy (3), we impose in (4)

$$e_1 + e_2 \exp\left(-j2\pi n \frac{k_2}{T}\right) = 0. \tag{5}$$

Since we impose real values to e_i, (5) holds only when $\exp(-j2\pi n k_2/T) = \pm 1$. Then, depending on the values e_i, (5) is verified by two cases. One is when $e_1 = e_2$ and always when $\exp(-j2\pi n k_2/T) = -1$. Thus, periodic gaps occur at positions n when $2nk_2/T = (2m + 1)$ is valid (i.e., observing the restrictions $0 < k_2 < T$ and n integer), where $m = 0, \pm 1, \pm 2, \cdots$. The other case takes place when $e_1 = -e_2$ and $\exp(-j2\pi n k_2/T) = 1$, giving $2nk_2/T = 2m$. In this case, since $0 < k_2 < T$, $m = 0$ implies a gap at $n = 0$ for any delay k_2. This last condition is important and will be further discussed. On both the above cases, because the e_i's are real values, the gaps are placed symmetrically with respect to the $n = 0$ axis of the spectrum (4). To simplify the recovery of $s(t)$, the preferred case is the one that produces gaps at $n = \pm 1$ because the needed interpolator is a single lowpass filter. In both cases, regarding e_i, however, one can verify that there is a unique choice of k_2/T to produce gaps at $n = \pm 1$. That is, the gaps at $n = \pm 1$ (as well as at $n = \pm 3, \pm 5, \cdots$ because they are periodic) take place only when $e_1 = e_2$ and $k_2/T = 1/2$, which is a case of uniform sampling. When $e_1 = -e_2$, there is a gap at the origin for any delay, as mentioned. Excepting this gap at $n = 0$, the closest gaps to the origin can only occur at $n = \pm 2$ (and at $n = \pm 4, \pm 6, \cdots$) when $k_2/T = 1/2$, which is also uniform sampling. For other positions n, there is some freedom in the choice of the delay. In the case $e_1 = -e_2$, for example, gaps take place at $n = \pm 3$ (as well as at $n = 0, \pm 6, \pm 9, \cdots$) by choosing $k_2/T = 1/3$ or $k_2/T = 2/3$. These cases are not explored here.

Since only when $k_2/T = 1/2$ and $e_1 = e_2$ is it possible to recover $s(t)$ using a single lowpass filter, we discuss now the case $e_1 = -e_2$, where a gap occurs at $n = 0$ for any delay k_2. To simplify the following analysis, let $e_1 = 1$ and $e_2 = -1$. Under this condition, consider the spectrum $S_2(f)$ of (4) multiplied by an ideal spectral window $G(f) = T$ with passband $|f| \leq 1/T$. This filter selects the lower bands of the replicas at $n = \pm 1$. The resultant spectrum after filtering is

$$F(f) = S^-(f - 1/T)[1 - \exp(-j\alpha)] + S^+(f + 1/T)[1 - \exp(j\alpha)] \quad (6)$$

where $\alpha = 2\pi k_2/T$. Here , we can verify that if $s(t)$ is bandlimited to $B' < 1/T$, the lower bands of the replicas at $n = 1$ and $n = -1$ can occupy the frequency interval $[-1/T, 1/T]$ without spectral overlapping the replica at $n = 0$ (which was suppressed). The inverse Fourier transform of (6), written in a more convenient form, is

$$f(t) = 2c\left[s(t)\cos\left(2\pi\frac{1}{T}t - \beta\right) + \hat{s}(t)\sin\left(2\pi\frac{1}{T}t - \beta\right)\right] \quad (7)$$

where $\hat{s}(t)$ is the Hilbert transform of $s(t)$, $c = 2 \cdot \sin(\alpha/2)$, and $\beta = \arctan[\sin(\alpha)/(1-\cos(\alpha)]$. To recover $s(t)$, a frequency shifting followed by lowpass filtering is necessary, as follows:

$$s(t) = \left[f(t) \cdot \frac{1}{2c}\cos\left(2\pi\frac{1}{T}t - \beta\right)\right] * g(t) \quad (8)$$

where $g(t)$ is the inverse Fourier transform of a lowpass filter with passband $|f| \leq B'$. In this case, the average sampling rate is minimum because $\overline{f}_S = 2/T$, and $B \leq 1/T$. Notice that although this recovering method uses only two lowpass filters, a modulation process is necessary. Thus, this method can be easily implemented only in systems where an oscillator is available, as in weather radar and Doppler ultrasonic systems. For Doppler velocimetry, however, $F(f)$ contains all relevant information about $s(t)$, which in this case is the Doppler shifted signal.[2] That is, for these applications, the most relevant data are the mean velocities and the velocity distribution widths. These data can be digitally computed (estimated) directly from the nonuniformly sampled signal spectrum (6), i.e., neither signal reconstruction nor delay knowledge are necessary [1].

IV. Recovery of Nth-Order Sampled Signals

For Nth-order sampling the average sampling frequency is equal to the Nyquist frequency if $N/T = 2B$. Now we consider $N > 2$, $B > 1/T$ and $N/T = 2B$. Only the recovering of $s(t)$ by selecting the replica at $n = 0$ is discussed, because the filter needed is a low-pass one. In these conditions more than one consecutive gap must be produced at positions $|n| > 0$, giving rise to an empty region R in the spectrum (2) between the remaining replicas. This region R is the frequency range from $n = 0$ (the replica being selected) to the first remaining replica after consecutive gaps at a position n. For example, for gaps at $n = 1$ and $n = 2$, the frequency range separating the adjacent replicas at $n = 0$ and $n = 3$ is $R = 3/T$. In general, $R = (l + 1)/T$, where $l = 0, 1, 2, \cdots$ is the number of gaps. To avoid spectral overlapping between the bands of the adjacent replicas at the edges of R, the condition $R \geq 2B$ must be verified. Combining this condition with the other one above ($N/T = 2B$) gives the minimum number of gaps in R, as $l_{\min} = N - 1$. The condition $N - 1$ gaps, when applied in (3), yields the same number of complex equations with $2N - 1$ real variables $e_1, \cdots e_N$ and $k_2, \cdots, k_N$ with $k_1 = 0$. Once the solutions are found, it is also necessary to verify if the replica at $n = 0$ has a nonzero value or, alternatively, if it is necessary to impose one more equation to ensure a nonzero value for the replica at $n = 0$. That is, let the left part of (3) equal a nonzero value (and real because e_i's are real). It is known that trivial solutions for the required gaps are $k_i/T = (i-1)/N$ and e_i equal to a constant, i.e., uniform sampling (as verified in Section III for second-order sampling). For $N > 2$, the search for the existence (or nonexistence) of solutions other than the trivial ones leads to increasing algebraic difficulties. Hence, only the case $N = 3$ is discussed in the Appendix, where we show that the unique solution is also the trivial one (uniform sampling). An alternative procedure is discussed next to find e_i's for certain delays values k_i's for the cases $N = 3$ and $N = 4$, where the number of gaps is less than the minimum necessary, implying average sampling rates higher than minimum.

For third-order sampling, if only one gap at $n = 1$ is desired in the sampled signal spectrum (2), implying nonminimum sampling rate, and predetermining a nonzero value to the replica at $n = 0$ (equal to one), these restrictions can be written as

$$\begin{vmatrix} 1 & W_2^{-1} & W_3^{-1} \\ 1 & 1 & 1 \\ 1 & W_2^1 & W_3^1 \end{vmatrix} \cdot \begin{vmatrix} e_1 \\ e_2 \\ e_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \quad (9)$$

where $W_i^n = \exp(-jn\alpha_i)$, and $\alpha_i = 2\pi k_i/T$. We have added the first row in the square matrix in (9), imposing a gap at $n = -1$, only to allow this matrix to be invertible, thus facilitating computations. By using a similar procedure[3] as in the Appendix, one can verify that there are always e_1, e_2 and e_3 real for any distinct delays k_2 and k_3, i.e., $k_2/T, k_3/T \neq 1/2$. For example, by directly computing (9) with $k_2/T = 1/10$ and $k_3/T = 1/7$, we have $e_1 = 3.6954$, $e_2 = -10.8602$, and $e_3 = 8.1647$. To reconstruct $s(t)$, the unique filter needed is a lowpass one with passband $|f| \leq B'$. However, the signal bandwidth must be $B' < 1/T$ because there are no gaps at $n = \pm 2$, except if $k_2/T = 1/3$ and $k_3/T = 2/3$, as shown in the Appendix. Since for $N = 3$ the average sampling rate is $3/T$, this sampling rate is $3/2$ times higher than the Nyquist rate. The advantage of this method is the simplicity of the interpolation for any delay k_i.

For $N = 4$, to select the replica at $n = 0$ if only two gaps at $n = 1$ and $n = 2$ in (2) are desired (whereas for minimum sampling rate, three gaps are needed) and predetermining a nonzero value to the replica at $n = 0$, we have

$$\begin{vmatrix} 1 & W_2^{-1} & W_3^{-1} & W_4^{-1} \\ 1 & 1 & 1 & 1 \\ 1 & W_2^1 & W_3^1 & W_4^1 \\ 1 & W_2^2 & W_3^2 & W_4^2 \end{vmatrix} \cdot \begin{vmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}. \quad (10)$$

Similarly, as in previous cases, there are always real values for e_i if $k_4/T = 1 - k_2/T$ for $k_2/T, k_4/T \neq 1/2$ (distinct values) and $k_3/T = 1/2$ (other delays were not searched for). Directly computing

[2]The quadrature detected Doppler signals, after being range-gated, are often at a lowpass position.

[3]The procedure is to search for delays to allow e_i to be real.

(10), for example, with $k_2/T = 1/8$, $k_3/T = 1/2$ and $k_4/T = 7/8$, we have $e_1 = -0.3535$, $e_2 = 0.5$, $e_3 = 0.3536$, and $e_4 = 0.5$. The signal bandwidth, however, must be $B' < 3/2T$. In this case, a lowpass filter having passband B' reconstructs the original signal. Since for $N = 4$ the average sampling rate is $4/T$, it is then $4/3$ times higher than the minimum, and the main advantage in this case is also the simplified filter.

V. Conclusions

A higher order sampling pattern is suggested to allow the recovering of a lowpass signal $s(t)$ from its nonuniform samples using only lowpass filters. For second-order sampling, it is possible to recover, by using a lowpass filter, a signal $f(t)$ from its nonuniform samples obtained with minimum rate, such that $f(t)$ contains all relevant information about the original signal for spectral estimations, which is useful for Doppler velocimetry. It is also possible to recover the original signal from $f(t)$ using a frequency shifter and another lowpass filter.

For the higher order sampling cases $n = 3$ and $N = 4$, we showed that by applying appropriate sampling restrictions, the reconstruction of the original signal is possible by using only lowpass filters but at a higher sampling rate ($3/2$ and $4/3$ times the minimum, respectively).

Appendix

For $N = 3$, two gaps are needed at $n = 1$ and $n = 2$ to recover $s(t)$ sampled with minimum average sampling rate, as already shown in Section IV. Entering these conditions in (3) and writing it in a more convenient way, we have

$$\begin{vmatrix} 1 & W_2^1 & W_3^1 \\ 1 & W_2^2 & W_3^2 \end{vmatrix} \cdot \begin{vmatrix} e_1 \\ e_2 \\ e_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \tag{A.1}$$

where $W_i^n = \exp(-jn\alpha_i)$, and $\alpha_i = 2\pi k_i/T$ or $W \cdot e_i = 0$, where W is the 2×3 matrix in (A.1), and e_i is the amplitude vector. Let the matrix W be written as $W = \text{Re}(W) + j\text{Im}(W)$. For e_1, e_2 and e_3 to be real, it is necessary that $\text{Im}(W) \cdot e_i = 0$, or

$$\begin{vmatrix} -\sin(\alpha_2) & -\sin(\alpha_3) \\ -\sin(2\alpha_2) & -\sin(2\alpha_3) \end{vmatrix} \cdot \begin{vmatrix} e_2 \\ e_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}. \tag{A.2}$$

For distinct $\alpha_i \neq \pi$, for the determinant of the square matrix in (A.2) to be zero, it is necessary that

$$\frac{\sin(2\alpha_2)}{\sin(\alpha_2)} = \frac{\sin(2\alpha_3)}{\sin(\alpha_3)}. \tag{A.3}$$

Remembering that $0 < k_i < T$ or $0 < \alpha_i < 2\pi$, then (A.3) is always true when $\alpha_3 = 2\pi - \alpha_2$ for distinct α_i's, hence excluding $\alpha_2, \alpha_3 = \pi$. Solving (A.2) for these values, we find $e_2 = e_3 = e$. Taking these values into $\text{Re}(W)$, we have

$$\begin{vmatrix} 1 & 2\cos(\alpha_2) \\ 1 & 2\cos(2\alpha_2) \end{vmatrix} \cdot \begin{vmatrix} e_1 \\ e \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}. \tag{A.4}$$

The determinant of the square matrix in (A.4) is zero if $\cos(\alpha_2) = -\cos(2\alpha_2)$, and the unique possible angle is $\alpha_2 = 2\pi/3$ (for $\alpha_2 < \alpha_3$). Solving (A.4) for this α_2, we get $e_1 = e$. Therefore, gaps situated at $n = \pm 1$ and $n = \pm 2$ only occur if $k_2/T = 1/3$, $k_3/T = 2/3$, and $e_1 = e_2 = e_3$, which is a uniform sampling case.

Acknowledgment

The authors would like to thank Dr. P. A. Tonelli and Dr. P. S. P. da Silva, who provided helpful discussions. They also are grateful to the reviewers who suggested various improvements to the work in this correspondence.

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Corrections to "Zolotarev Polynomials and Optimal FIR Filters"

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I. Introduction

In the above paper,[1] the following corrections need to be made.

- Equation (21) should read

$$w_m = w_s + 2\frac{sn(u_0)cn(u_0)}{dn(u_0)} Z(u_0).$$

- The equations on p. 722 should read

$$b(3)\Delta = (w_s - w_p)(1 + w_s w_p) + (1 - w_p^2) - (1 - w_s^2)$$
$$b(2)\Delta = -w_s w_p (w_s^2 - w_p^2) - w_p(1 - w_p^2) + w_s(1 - w_s^2)$$
$$b(1)\Delta = w_p^3 - w_s^3 + w_s^2 w_p^2 (w_s - w_p) - (1 - w_p^2) + (1 - w_s^2)$$
$$b(0)\Delta = w_s w_p (w_s^2 - w_p^2) + w_p(1 - w_p^2) - w_s(1 - w_s^2).$$

- In Tables IV and V, instead of

$$w_m = w_s + 2\frac{dn(u_0)}{sn(u_0)cn(u_0)} Z(u_0) \tag{19}$$

there should be

$$w_m = w_s + 2\frac{sn(u_0)cn(u_0)}{dn(u_0)} Z(u_0). \tag{21}$$

- In the design procedure, the equation under item 5 on p. 727 should read

$$w_m = \cos\omega_m T = w_s + 2\frac{sn(u_0)cn(u_0)}{dn(u_0)} Z(u_0).$$

Manuscript received August 17, 1999; revised September 6, 1999. The associate editor coordinating the review of this paper and approving it for publication was Editor-in-Chief JoséM. F. Moura.

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Publisher Item Identifier S 1053-587X(00)04947-3.

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