THE F_N METHOD IN RADIATIVE TRANSFER AND NEUTRON TRANSPORT THEORY

J. R. Maiorino
Centro de Engenharia Nuclear
Instituto de Pesquisas Energéticas e Nucleares
São Paulo, S.P., Brazil

The F_n method basic to radiative transfer and neutron transport theory is discussed. In order to illustrate the accuracy of the method, a numerical example of the albedo and transmission factor of the slab albedo problem in one speed anisotropic transport theory is given and compared with "exact" results .

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INTRODUCTION

Recently, Siewert and coworkers[1,2], introduced a new approximated method, called the F_n method (the capital letter F stands for the French word "facile". The F_n method follows closely the early C_n method of Benoist[3], but it yields more concise equations that can be solved numerically even more efficiently than the C_n method. The F_n method, uses partially Case's technique[4] in order to derive a set of singular integral equations for the angular distributions at the boundaries, and then approximates these angular distributions by a polynomial of order n to derive a set of linear algebraic equations for the coefficients of the polynomial approximation. In short the F_n method can be summarized in the following steps :

i) for each specific problem we write the transport equation and the appropriate boundary conditions (usually conditions for the angular distributions at the boundaries) .

ii) write the "formal" solution as given by Case's technique, i.e, the normal mode expansion .

iii) derive singular integral equations relating the angular distributions at the boundaries by using the full-range orthogonality properties of the Case's eigenfunctions .

iv) approximate the angular distributions at the boundaries by a polynomial of order N .

v) substitute the approximation, described in iv, into the singular integral equations, derived in iii, then apply the boundary conditions and obtain a system of linear algebraic equations for the coefficients of the polynomial approximation .

Up to present, we applied the F_n method to various problems in radiative transfer and neutron transport theory in order to extend the range of possible applications, and to demonstrate the numerical accuracy of the method. We have applied the F_n method together with "exact" analysis to find the radiation field due to a point source of radiation located at the center of a finite sphere [5]. By solving this problem we have shown that the F_n method can provide very accurate numerical results, since we have achieved excellent agreement with "exact" results using low orders of the F_n approximation. Besides, the "exact" solution of this problem requires the iterative numerical solution of a set of Fredholm integral equations, whereas the F_n method requires only the numerical solution of linear algebraic equations. We also have compared the F_n results with a Monte Carlo results, and note that to obtain the same degree of accuracy, as the F_n method, the Monte Carlo requires a large number of histories .

We also considered the application of the F_n method to a radiative transfer problem in an anisotropically scattering plane parallel medium with specularly and diffusely reflecting boundaries, and internal heat sources[6]. Numerical results for quantities basic to compute the net radiative heat flux in a Mie scattering media were found .

We considered basic problem in reactor physics, such as the critical problem for multiregion reactors, and the thermal disadvantage factor calculation basic in the study of thermal utilization in heterogeneous reactor cells[7,8]. We found numerical results for the critical half - thickness for various test problems, as well as the disadvantage factor for various heterogeneous cells which are in perfect agreement with those reported in the literature. Another type of problem considered was the application of the F_n method for solving a classical problem, basic to astrophysics and atmospherical science, namely the transfer of polarized light in a mixture of Rayleigh and isotropic scattering atmosphere, with ground reflection, and considering zenithal and azimuthal dependence, the so-called "planetary problem"[9,10]. We note that this problem is a transport problem in multidimension geometry .

In order to illustrate the accuracy of the F_n method, we give a numerical example of the albedo and transmission factor of the slab albedo problem in one speed anisotropic transport theory, and compare our results with the "exact" results of Kaper et al[11].

SLAB ALBEDO PROBLEM-ANALYSIS

As an illustration of the F_n method, we now consider the slab albedo problem in one speed anisotropic transport theory, and thus we seek solution of the equation .

$$\mu \frac{\partial}{\partial x} I(x,\mu) + I(x,\mu) = \frac{\omega}{2} \sum_{\ell=0}^{L} (2\ell+1) f_\ell P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu') I(x,\mu') d\mu' , \tag{1}$$

subject to the boundary conditions

$$I(0,\mu) = \frac{1}{2\pi\mu_0} \delta(\mu-\mu_0) , \qquad \mu, \mu_0 > 0 \tag{2-a}$$

and

$$I(\tau,-\mu) = 0, \ \mu > 0 \ . \tag{2-b}$$

Here, $I(x,\mu)$ is the angular flux, ω is the single scattering albedo, or the number of secondary neutron in neutron transport theory, f_ℓ are the coefficients in a Legendre expansion of the phase function, or transfer function, μ is the direction cosine, μ_0 is the direction cosine of the beam incident upon the free surface x=0 , τ is the slab optical thickness and $P_\ell(\mu)$ are the Legendre polynomials .

The solution of equations (1) can be written in terms of Case's eigenfunctions[4]

$$I(x,\mu) = \sum_{i=0}^{\kappa-1} \left\{ A_1(\nu_i) \phi(\nu_i,\mu) e^{-x/\nu_i} + A(-\nu_i) \phi(-\nu_i,\mu) e^{x/\nu_i} \right\} + \int_{-1}^{1} A(\nu) \phi(\nu,\mu) e^{-x/\nu} d\nu ,$$

where κ is the number of pairs of eigenfunction of

$$\Lambda (Z) = 1 + Z \psi (Z) \log \left\{ \frac{Z-1}{Z+1} \right\} + \omega Z \sum_{\ell=1}^{L} (2\ell+1) f_\ell \, g_\ell (Z) \, \Gamma_\ell (Z) = 0 \; . \quad (4)$$

Here, the functions $g_\ell (Z)$, $\Gamma_\ell(Z)$ are defined[6] by :

$$(2\ell+1) \, Z \, \Gamma_\ell (Z) = - \delta_{\ell,0} + (\ell+1) \, \Gamma_{\ell+1} (Z) + \ell \, \Gamma_{\ell-1} (Z) \; , \quad (5\text{-a})$$

with

$$\Gamma_o (Z) = 0 \quad (5\text{-b})$$

and

$$(2\ell+1) \, (1-\omega f_\ell) \, \nu \, g_\ell (\nu) = (\ell+1) \, g_{\ell+1} (\nu) + \ell \, g_{\ell-1} (\nu) \quad (6\text{-a})$$

with

$$g_o (\nu) = 1 \quad \text{and} \quad g_1 (\nu) = \nu \, (1 - \omega) \quad (6\text{-b})$$

In addition

$$\psi (Z) = \frac{1}{2} \, \omega \sum_{\ell=o}^{L} (2\ell+1) \, f_\ell \, g_\ell (\nu) \, P_\ell (\mu) \; , \quad (7)$$

and the eigenfunctions, $\phi (\xi,\mu)$, $\xi = \pm \nu_i$ ou $\in (-1,1)$, are those reported in reference 6, and so there is no need to repeat here all the required definitions .

Now, we use the full-range orthogonality relations concerning the functions $\phi (\xi,\mu)$ to develop a system of singular integral equations for the angular fluxes at the surface of the slab, i.e.

$$\int_0^1 \mu \, \phi (\xi,\mu) \, I (0,-\mu) d\mu + e^{-\tau/\xi} \int_0^1 \mu \, \phi (-\xi \, \mu) \, I (\tau,\mu) d\mu = L_1 (\xi) \quad (8\text{-a})$$

and

$$\int_0^1 \mu \, \phi (\xi,\mu) \, I (\tau,\mu) \, d\mu + e^{-\tau/\xi} \int_0^1 \mu \, \phi (-\xi,\mu) \, I (0,-\mu) d\mu = L_2 (\xi) \, , \quad (8\text{-b})$$

where

$$L_1 (\xi) = \frac{1}{2\pi} \, \phi (-\xi, \mu_o) \quad (9\text{-a})$$

and

$$L_2 (\xi) = \frac{1}{2\pi} \, \phi (\xi,\mu_o) \, e^{-\tau/\xi} \; . \quad (9\text{-b})$$

We note that to derive equations (8) we have used the boundary conditions, as given by equations (2), and that in equations (8), $\xi \in P$, where $P = \{\nu_\beta\} \cup (0,1)$. We now introduce the F_n approximations .

$$I (0,-\mu) = \sum_{\alpha=o}^{N} a_\alpha \, \mu^\alpha \quad (10\text{-a})$$

and

$$I (\tau, \mu) = \frac{1}{2\pi\mu_o} \, \delta (\mu - \mu_o) \, e^{-\tau/\mu_o} + \sum_{\alpha=o}^{N} b_\alpha \, \mu^\alpha \; , \quad (10\text{-b})$$

into equations (8), to derive the F_n equations

$$\sum_{\alpha=0}^{N} A_\alpha B_\alpha (\xi) + e^{-\tau/\xi} \sum_{\alpha=0}^{N} b_\alpha A_\alpha (\xi) = \frac{g(-\xi,\mu_o)}{2\P(\xi+\mu_o)} \left\{ 1 - e^{-\tau\,(1/\xi + 1/\mu_o)} \right\},$$

$$\xi \in P, \qquad (11\text{-a})$$

and

$$\sum_{\alpha=0}^{N} b_\alpha B_\alpha (\xi) + e^{-\tau/\xi} \sum_{\alpha=0}^{N} a_\alpha A_\alpha (\xi) = \frac{g(\xi,\mu_o)}{2\P} \left\{ \frac{e^{-\tau/\xi} - e^{-\tau/\mu_o}}{\xi - \mu_o} \right\},$$

$$\xi \in P, \qquad (11\text{-b})$$

where

$$g(\xi,\mu) = \sum_{\ell=0}^{L} (2\ell+1)\, f_\ell g_\ell (\nu)\, P_\ell (\mu) . \qquad (12)$$

In additions, the F_n functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ are difened by

$$A_\alpha (\xi) = \frac{2}{\omega\,\xi} \int_0^1 \mu^{\alpha+1}\, \phi(-\xi,\mu)\, d\mu \qquad (13\text{-a})$$

and

$$B_\alpha (\xi) = \frac{2}{\omega\,\xi} \int_0^1 \mu^{\alpha+1}\, \phi(\xi\mu)\, d\mu , \qquad (13\text{-b})$$

and can be obtained by the recurrence relation discussed in reference 6 .

To find the coefficients a_α and b_α, $\alpha = 0,1,2, \ldots N$, we can select (N+1) values of $\xi \in P$ to derive a set of 2 (N+1) linear algebraic equations which can be solved by the usual numerical techniques. Once a_α and b_α are established, the exit distribution $I(0,-\mu)$ and $I(\tau,\mu)$, $\mu > 0$, can be determined by using equations (10). Also, other quantities of physical interest, such as the albedo and the transmission factor, are readily available. For example, the albedo .

$$A = 2\P \int_0^1 \mu\, I(0,-\mu)\, d\mu , \qquad (14)$$

and the uncollided transmission factor

$$T = 2\P \int_0^1 \mu\, I(\tau,\mu)\, d\mu - e^{-\tau/\mu_o} \qquad (15)$$

can be expressed, in the F_n approximation, by

$$A = 2\P \sum_{\alpha=0}^{N} (a_\alpha / \alpha + 2) , \qquad (16)$$

and

$$T = 2\P \sum_{\alpha=0}^{N} (b_\alpha / \alpha + 2) . \qquad (17)$$

Finally, the complete solution of this problem can be obtained by the method described in reference (5) .

SLAB ALBEDO PROBLEM-NUMERICAL RESULTS

For our numerical calculation we consider the same scattering law used by Kaper et al [11], in order to have a standard of comparison. To solve the linear algebraic system of equations, we have selected $\xi_\beta = \nu_\beta$, $\beta = 0, 1, \ldots, \kappa - 1$, and the remaining ξ_β as given by

$$\xi_{j+\kappa-1} = \frac{2j-1}{2(N-\kappa+1)}, \quad j = 1, 2, \ldots, (N-\kappa+1), \tag{18}$$

Where N is the order of approximation, and ν_β are the discrete eigenvalue and can be found by an iterative solution of equation (4) .

In order to illustrate the accuracy of the F_n method we give in Table 1, a numerical example of the albedo and transmission factor and compare with the "exact" results of Kaper et al [11]. In addition, in Table 2 we give the exit angular distributions computed by the F_n method, and note that it is in perfect agreement with that of Kaper .

CONCLUSIONS

From the problems we solved we have concluded the following advantages of the F_n method :

i) the mathematical analysis is simple, and does not require any special technique .

ii) the computational requirements are simple, since the method needs only the computation of simple functions, and the solution of a linear algebraic system of equations. Also with exception of the critical problem, the F_n method does not require any iterative numerical methods .

iii) the method uses the actual boundary conditions, contrary to other methods, like P_n method in which is necessary to use non-physical boundary conditions. Also, the method only approximates unknown quantities at the boundaries .

On the other side, some disadvantages inherent to the F_n method are :

i) the choice of the points to satisfy the F_n equations, will affect the way the results converge .

ii) the accuracy of the method is presently limited, due to numerical reasons, to 4 or 5 figures, and even using higher order approximation we could not improve this accuracy .

Finally, we should mention that we are doing some additional work in order to extend the method in solving pratical problems. For example, we will try to aplly the method for solving the energy dependent transport equation, mainly the gamma ray transport equations, with application in shielding . Also, studies in other geometries, and in multidimension geometries need to be considered. In addition, transport problem in other areas, like kinetic theory, may be capable of treatment by the F_n method .

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Table 1 - Albedo and Transmission factor of the Slab Albedo Problem in One-Speed Anisotropic Transport Theory[a]

	Degree of anisotropy			Degree of anisotropy		
	F_3	F_5	"Exact"	F_3	F_5	"Exact"
Albedo	0.3205	0.3209	0.32092	0.6968(-1)	0.7783(-1)	0.77125(-1)
Transmission	0.4396	0.4392	0.43920	0.6811	0.6702	0.67002

Table 2 - Converged F_N results for the Angular distribution at the boundaries ($\tau = o$ and $\tau = \tau o$). Degree of anisotropy is 2

μ	$I(o,-\mu)$	$I(\tau o,\mu)$
0.0	0.7092	0.2777
0.1	0.6626	0.3439
0.2	0.5970	0.4071
0.3	0.5258	0.4463
0.4	0.4570	0.4631
0.5	0.3942	0.4654
0.6	0.3384	0.4597
0.7	0.2891	0.4497
0.8	0.2455	0.4376
0.9	0.2068	0.4246
1.0	0.1721	0.4109

[a]The phase function for this problem is the same as that used by Kaper et all[11], the single scattering albedo is ω=0.95, the slab thickness is τ=1, and the cosine of the incident angle is μo=0.5 .