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Hierarchical expansion method in the solution of the Navier–Stokes equations for incompressible fluids in laminar two-dimensional flow

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ABSTRACT

This work presents a hierarchical expansion method applied in the solution of the Navier–Stokes equations for incompressible fluids in laminar two-dimensional flow. The expansion functions used are based on Legendre polynomials, adjusted in such a way that the order of the expansion can be adjusted to the necessary order. The results show the capability of the method to yield accurate results.

Keywords: **Finite** element, Petrov–Galerkin formulation, Navier–Stokes equation, Hierarchical expansion functions, Incompressible fluid

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Abstract

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1. Introduction

The objective of this work is to develop and to apply the method of hierarchical expansion, proposed by Zienkiewicz and Morgan [1], in the solution of the two-dimensional Navier–Stokes equations, for incompressible fluids in laminar flow. This method is based on the finite element method using the Petrov–Galerkin formulation [2]. All variables describing the fluid flow are expanded in nearly hierarchical functions based on Legendre polynomials. This method has some advantages over other numerical schemes as is described in the following sections.

2. Conservation equations (2D)

The equations governing the fluid dynamics are continuity, momentum and energy equations. In this work, the method proposed by Chorin [3] is used for the treatment of the coupling between pressure and velocity. The following equations are used with this method:

$$\rho\left(\frac{u^* - u^{t-\Delta t}}{\Delta t}\right) + \rho u\frac{\partial u^*}{\partial x} + \rho w\frac{\partial u^*}{\partial z} - \mu\left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial z^2}\right) = 0; \quad (1)$$

$$\rho u = \rho u^* - \Delta t\frac{\partial p}{\partial x}; \quad (2)$$

$$\rho\left(\frac{w^* - w^{t-\Delta t}}{\Delta t}\right) + \rho u\frac{\partial w^*}{\partial x} + \rho w\frac{\partial w^*}{\partial z} - \mu\left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial z^2}\right) = 0; \quad (3)$$

$$\rho w = \rho w^* - \Delta t\frac{\partial p}{\partial z}; \quad (4)$$

$$\nabla^2 p = \frac{\rho}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial z}\right); \quad (5)$$

where ρ is the fluid specific mass, t is the time, Δt is the time step, u and w are the velocity components in the x and z directions, respectively, u^* and w^* are the

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intermediate values of the velocity components in the x and z directions respectively, p is the fluid pressure, μ is the fluid dynamic viscosity, and the superscript $t-\Delta t$ refers to the variables calculated at the previous time step. To make the nomenclature easier, no superscript is used to denote the variables at the present time.

Assuming constant fluid properties, the energy equation written in terms of the temperature for Cartesian coordinates in two dimension is the following:

$$\rho\frac{\partial T}{\partial t}+\rho u\frac{\partial T}{\partial x}+\rho w\frac{\partial T}{\partial z}=\frac{k}{c_p}\nabla^2 T+\frac{\mu}{c_p}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \frac{2\mu}{c_p}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right], \tag{6}$$

where T is the temperature, k is the thermal conductivity and c_p is the specific heat at constant pressure.

3. Mathematical development of the conservation equations for a single element

A rectangular structure mesh is used to divide the flow domain into nodes. Although this is the simplest form of grid it is enough to show the capabilities of the proposed method.

For each node, Eqs. (1), (2), (3), (4), (5) and (6) are multiplied by a weighting function and integrated. The Green's Theorem is used to simplify the diffusive terms in Eqs. (1), (3), (5) and (6).

The weighting function, P_m, is based on the Petrov–Galerkin formulation and it is given, according to Brooks and Hughes [2], by:

$$P_m = N_m + \frac{\Delta t}{2}\overline{u}_{i,j}\frac{\partial N_m}{\partial x} + \frac{\Delta t}{2}\overline{w}_{i,j}\frac{\partial N_m}{\partial z}, \tag{7}$$

where $\overline{u}_{i,j}$ and $\overline{w}_{i,j}$ are, respectively, the average velocity in the x and z directions in node i, j, and N_m is the mth expansion function for node i, j.

For each node, the variables u^*, w^*, u, w, p and T, are expanded in a series as follows:

$$u^* = \sum_{m=1}^{M} u_m^* N_m; \quad w^* = \sum_{m=1}^{M} w_m^* N_m; \; u = \sum_{m=1}^{M} u_m N_m; \quad w = \sum_{m=1}^{M} w_m N_m; \; p = \sum_{m=1}^{M} p_m N_m; \quad T = \sum_{m=1}^{M} T_m N_m; \tag{8}$$

where the parameters u_m^*, w_m^*, u_m, w_m, p_m and T_m are the coefficients of the variables associated with the m^{th} expansion function for that node.

For example, this process applied to the pressure equation, Eq. (5), results in the following expression for node i, j:

$$\sum_{n=1}^{M} p_n \left\{ \frac{\Delta z_{i,j}}{\Delta x_{i,j}} \left[\int_{-1}^{1} \left(N_m \frac{\partial N_n}{\partial \xi}\right)_{\xi=-1}^{\xi=1} \mathrm{d}\eta - \int_{-1}^{1}\int_{-1}^{1} \frac{\partial N_m}{\partial \xi}\frac{\partial N_n}{\partial \xi}\,\mathrm{d}\xi\,\mathrm{d}\eta \right] - \frac{\Delta x_{i,j}}{\Delta z_{i,j}} \left[\int_{-1}^{1} \left(N_m \frac{\partial N_n}{\partial \eta}\right)_{\eta=-1}^{\eta=1} \mathrm{d}\xi - \int_{-1}^{1}\int_{-1}^{1} \frac{\partial N_m}{\partial \eta}\frac{\partial N_n}{\partial \eta}\mathrm{d}\xi \mathrm{d}\eta \right] \right\}$$
$$= \frac{\rho \Delta z_{i,j}}{2\Delta t} \sum_{n=1}^{M} u_n^* \int_{-1}^{1}\int_{-1}^{1} P_m \frac{\partial N_n}{\partial \xi}\,\mathrm{d}\xi\,\mathrm{d}\eta - \frac{\rho \Delta x_{i,j}}{2\Delta t} \sum_{n=1}^{M} w_n^* \int_{-1}^{1}\int_{-1}^{1} P_m \frac{\partial N_n}{\partial \xi}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$
$$\text{for } m = 1, \ldots, M. \tag{9}$$

where $\Delta x_{i,j}$ and $\Delta z_{i,j}$ are the length of node i, j in the x and z direction, respectively, ξ and η are the node locally space variables in the x and z directions, respectively. This expression represents a system of M equations, where M is the total number of expansion functions used.

In matricial form, this system of equations is written as follows:

$$\mathrm{A}_p^{i,j}\mathbf{p} = \mathrm{b}_p^{i,j}, \tag{10}$$

where $\mathrm{A}_p^{i,j}$ is the pressure equation matrix for node i, j, which contains all the right hand side terms of Eq. (9), $\mathbf{p}$ is the vector of the pressure expansion coefficients, and $\mathrm{b}_p^{i,j}$ is a vector for node i, j which contains the left hand side terms of Eq. (9). For Eqs. (1), (3) and (6), the coefficient matrices and the left hand side vectors are functions of the velocity expansion coefficients.

After applying this process for all equations at every node of the mesh, a system of equations results which must be solved interactively at each time step to calculate the expansion coefficients.

4. Hierarchical functions

In the classic finite element method the expansion coefficients are identified with the variables at specified locations. This identification has been widely followed in the finite element literature and has the merit of assigning a 'physical' meaning to these coefficients. There is, however, a great disadvantage in this practice. If it is desired to change the order of the expansion, it is necessary to restart the problem due to the complete change of all expansion functions involved [1].

In the case of the hierarchical expansion functions, the expansion coefficients are not identified with the physical variables at specific points of the mesh. In this case, the coefficients are associated with the expansion functions, which are adjusted in the rectangular nodes, defining corner, side and area functions. This association allows to start the solution of a problem with a linear expansion and if necessary during the solution process, to add new shape functions to increase the order of expansion and to obtain a more accurate solution. In this case, the expansion functions do not change by adding or deleting new expansion functions to change the order of the expansion. Thus, it is not necessary to recalculate the matrices. Changing the order of expansion without the need to restarting the problem turns out to be the great advantage of this method.

The hierarchical expansion functions are based on Legendre polynomials, which form a set of functions with orthogonality properties. Due to the association of two functions in each direction to form the two-dimensional expansion functions, complete orthogonality is impossible to achieve, but the main diagonal terms in the matrices are dominant.

Fig. 1 shows four nodes and the parameters associated with their corners, sides, and areas. In this figure, $\phi^c_{i,j}$ are the corner parameters associated with the corner (linear) expansion functions, $\phi^x_{i,j,g}$ and $\phi^z_{i,j,g}$ are the side parameters associated with the side expansion functions, and $\phi^A_{i,j,g}$ are the area parameters associated with the area expansion functions. The order of the approximation for the side and area expansion functions is up to the desired or necessary degree.

Each corner parameter is connected to four elements through four different expansion functions and each side parameters is connected to two elements by two different functions. The area parameters belong to a single element only. The association of the corners and side expansion coefficients with adjacent elements guarantees the singularity and the continuity of the solution at all node boundaries.

It should be observed that it is very easy to increase the order of expansion locally to achieve a refinement in the region where the parameters vary most rapidly and where the approximation is therefore prone to the largest errors.

5. Results

In order to validate the numerical method proposed in this work, some well-known problems of the literature are

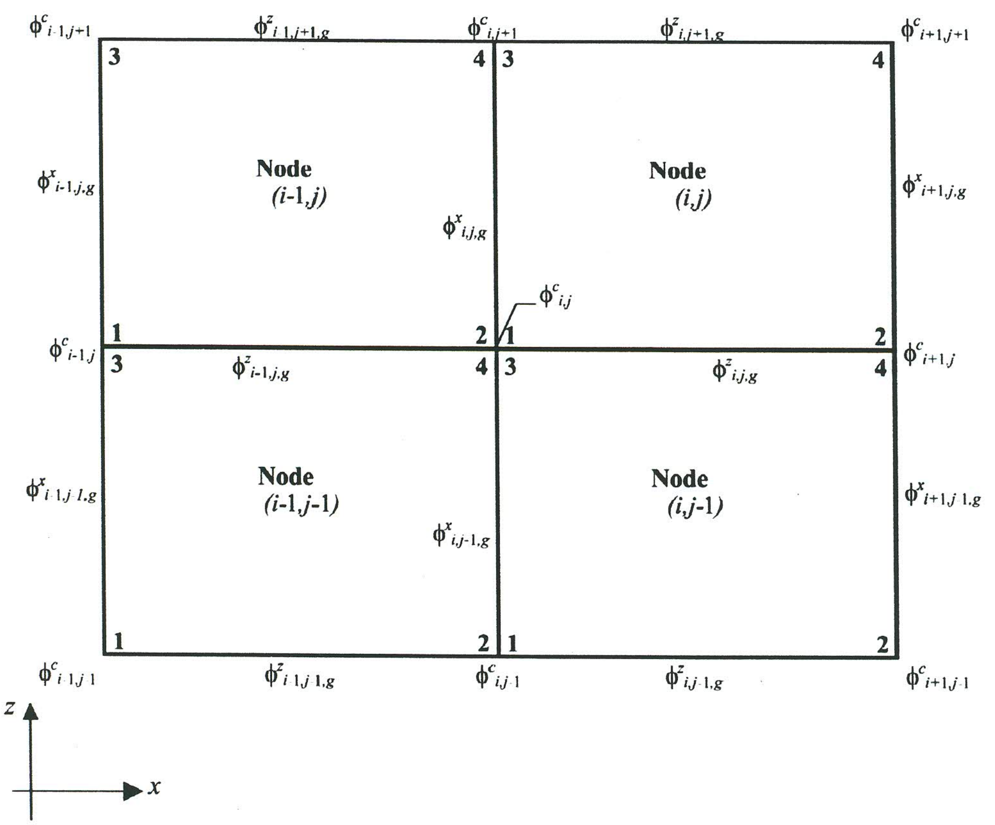

Fig. 1. Four rectangular grid nodes and associated parameters.

Table 1
Mean error between the calculated and experimental results for the velocity

Position x (m)	Mean error degree 1 (%)	Mean error degree 2 (%)	Mean error degree 3 (%)
0.012	8.70	4.42	3.01
0.030	10.3	8.3	4.19
0.060	13.47	11.95	7.09
0.090	13.653	12.51	8.17
0.12	15.92	13.21	10.29

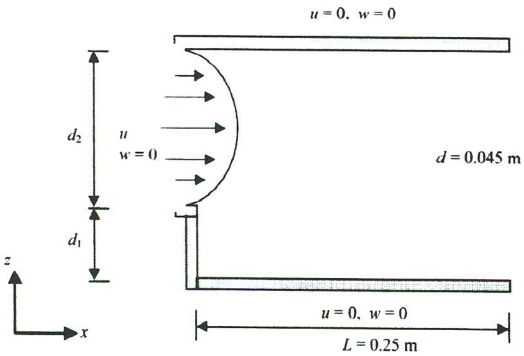

Fig. 2. Backward-facing step geometry.

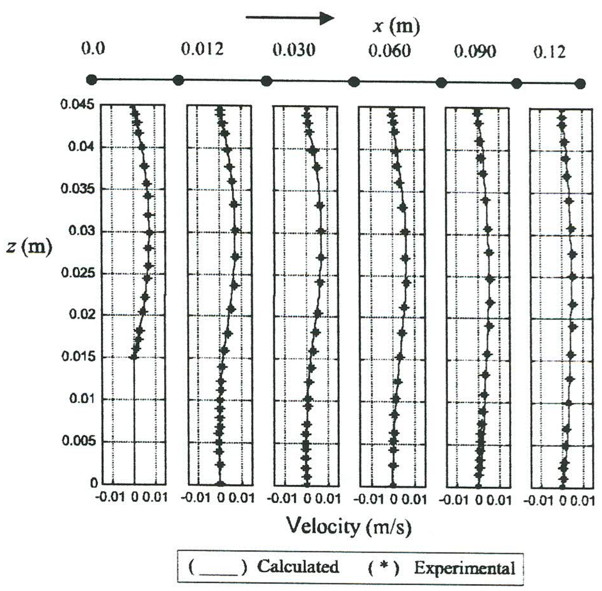

Fig. 3. Calculated results (third order expansion) and experimental results from Denham and Patrick experiment with Reynolds = 73 [5].

simulated. This paper presents the results of a backward-facing step problem, additional results can be found in Sabundjian [4].

For the accomplishment of this analysis, the experimental results obtained by Denham and Patrick [5] are used. Fig. 2 shows the geometry of the problem. A course mesh with 25 × 12 elements and expansion orders 1, 2 and 3 are used. Fig. 3 presents graphical results for the velocity for the case of third order expansion. Table 1 presents the mean errors between the calculated and the experimental results. As the expansion order increases, the calculated results are closer to the experimental results. However, the differences between the numerical and experimental results increase with the distance from the inlet.

6. Conclusions

Based on the results obtained in the solution of the proposed problems, it is possible to conclude that the method of hierarchical expansion is suitable for solution of incompressible fluid problems in two dimensions. The calculated results are in good agreement with the experimental results. The great advantage of the hierarchical expansion method is the capacity to adapt the expansion order to the necessary or desired value during the flow calculation, without the necessity to restart the problem, as happens in the conventional finite element method, or in the method of mesh refinement.

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