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ANALYSIS OF NON-LINEAR STRUCTURES SUBJECTED TO

DYNAMIC LOADING, USING DYNAMIC RELAXATION

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SUMMARY

The particular method of Dynamic Relaxation which has been developed at IFA is very simple. The structure is divided into blocks and the displacements are repeatedly calculated from the stresses, and vice-versa, using simple finite difference equations.

There are some important advantages when dealing with non-linear structures. Because of the iterations, non-linear stress-strain relations can be applied very easily and accurately. Also, because there is complete freedom to change the boundaries during the iterations, cracks can be correctly represented, and will grow compatibly with the structure and the loads.

The above programs, originally written for static loads, can be applied directly to evaluate the effects of dynamic loading. The same iterations which are being used to analyse the dynamic behaviour of the structure, can be used simultaneously to input the changing loads.

Some results have already been published for a prestressed concrete cylindrical vessel with an internal explosion. Further work is being carried on with much simpler non-linear structures for which classical analysis can give some approximate solutions, in order to validate the method.

1. INTRODUCTION.

1.1. The advent of nuclear power has required the structural engineer to exercise skills of a higher order than had previously been generally available. In the case of the authors, the problem concerned the design of prestressed concrete pressure vessels for nuclear reactors. These are not only very thick redundant structures, but it was necessary to continue the analysis through stages of overload, with cracking of the structure, up to ultimate failure.

1.2. The basic purpose of the analysis was, of course, to permit such structures to be designed so that they would meet all requirements of integrity, as laid down in the specifications. That is to say, the analysis had to be very simple and cheap, as well as accurate, so that the designs could be prepared by trial and error methods. Similar considerations apply in many other fields, which are not confined to nuclear structures, and the methods to be described have wide application.

1.3. Although the method of Dynamic Relaxation is well understood and widely used in Europe, it has made slight headway on this side of the Atlantic. It is indeed surprising that this should be the case, as the advantages are substantial. Additionally, the basic methods have been developed at the Institute of Atomic Energy, Sao Paulo, in a way which makes the programs even more versatile and convenient. There are three of these programs: PV1 deals with two-dimensional structures in plane stress; PV2 analyses three-dimensional structures which have a polar axis of symmetry; PV3 can analyse general three-dimensional structures, described in cylindrical co-ordinates.

2. BASIC DYNAMIC RELAXATION.

2.1. The purpose of the analysis is to calculate the stresses and deflec-

tions in every part of the structure, however loaded. The method to be described will not only do this, but it also has the merit of great simplicity. The authors feel that simplicity is a most desirable attribute, making the methods available to any engineer. Therefore the whole approach is to retain the utmost simplicity, even at the expense of slightly uneconomic use of the computer.

2.2. The structure is first divided into as many imaginary blocks as are required to provide a sufficiently comprehensive analysis, remembering that one set of orthogonal stresses and deflections will be calculated in each block. In order to do this calculation, simple finite-difference equations are set up in each block which relate its stresses to the deflections in it, and in the neighbouring blocks. This collection of simultaneous equations is then solved by a process of successive approximation, which has been given the name Dynamic Relaxation, to provide the required answers.

2.3. There is a very simple physical analogy to Dynamic Relaxation, and when this has been mastered the method can be applied correctly, without troubling to understand the underlying mathematics. Imagine the structure, which is now represented by an array of imaginary blocks, to be at rest with zero stresses and deflections. Now apply the loads suddenly and examine each block in turn; by applying Newton's Law we can calculate the acceleration of each block to which an external load is applied; if we now choose some short interval of time, we can calculate the deflection of that accelerating block at the end of the time interval. We finish up with an array of deflections. We now re-examine every block in turn, and where there is a deflection we apply the above noted finite difference equations to calculate a stress, finishing up with an array of stresses. This completes the first iteration.

2.4. Subsequent iterations simply repeat the above procedure, but now the accelerating forces will include the stresses which have just been calcula-

ted, in addition to the external loads. In order to bring the vibrating structure to rest in the shortest possible time a viscous damping is applied to the accelerations. It will be apparent that at every iteration the external loads are in equilibrium with the internal stresses plus the inertia loads, and when the structure comes to rest the inertia loads are zero. Also the internal stresses are always compatible with the deflections. Therefore the values of the stress and deflection arrays when the structure comes to rest will then constitute the correct solution. It is somewhat paradoxical that one can best analyse a static structure with static loads by treating it as though it were dynamic. In fact, if the time interval and damping coefficient are well chosen, the method is very efficient, as well as simple.

2.5. In practice it is more convenient to work in terms of velocities, rather than accelerations, so that only first order finite difference equations are required. Then one alternately calculates velocities in terms of loads plus stresses, and stresses in terms of velocities, with subsidiary calculation of deflections in terms of velocities. In order to show the great simplicity of the method, an explanation of the stresses and deflections in a two-dimensional block has been included in the Appendix, together with the derivation of the basic equations, and a flow-chart of the program.

2.6. In order to apply the method it is necessary to know how to choose appropriate values of time interval and damping coefficient, and how to ensure that the relaxation has been continued for a sufficient number of iterations. These points have been covered in references 1 and 2.

3. BOUNDARIES.

3.1. Where a boundary forms one side of a block special equations will be required in two respects:- the shear may be zero, and the velocity equation for that side must consider the actual forces applied, and the fact that this is only a half-block. There is absolutely no difficulty in deriving

appropriate equations, analogous to those shown in the Appendix. The procedure which the authors have adopted is to allocate a code number to each such special block. Then when the structure is being divided up into blocks, before the analysis, the appropriate code number can be given to each block. This array of code numbers is then input to the program, so that appropriate equations will be selected by the computer. There are between 50 and 100 special blocks available in each program, and more can readily be added if required: the special blocks include not only every possible arrangement of orthogonal boundaries, including cracks (to be described later), but also diagonal boundaries, and those composed of arcs of circles. A simple but extremely useful special block has all its parameters zero: this can be used to fill out portions of the array where there is no structure, as for instance the internal cavity of a pressure vessel.

3.2. There is also provision in the programs to vary the size of the blocks, provided of course that the dimension in any row or column of the array is constant. This means that the mathematical model which is going to be analysed can always be made to agree with the actual structure. There is thus great adaptability in the application of the programs, and this feature can be used very easily.

4. CRACKS.

4.1. The freedom and ease with which boundaries can be described and calculated mean that with these programs cracks can be treated in the most accurate manner, i.e. by introducing two new boundaries which constitute the two sides of each crack. The only limitation is that the cracks must run between blocks, which imposes a zig-zag path, rather than a diagonal one. By choosing a suitably small block size the resulting errors can be made insignificant. The appropriate equations to calculate the deflections of each side of any crack can be derived just as easily as those in the Appendix: it is then a simple matter to calculate the width of any crack at each block.

4.2. In order to determine the length of each crack which is compatible with the applied loads, it has been found that any examination of stresses near the crack tip provides an unreliable criterion, because the stress contours are steep, and the material is not linearly elastic, at that point. Instead it has been found preferable to use a criterion from fracture mechanics, the crack opening displacement. This can be applied with ease, because it is only necessary to examine the width of the crack at the block next to the tip. If this is more than a certain amount, then the crack must be allowed to extend.

4.3. It has been found that it is quite safe to make this check after about half the required number of iterations has been performed. If the width exceeds the criterion, the program changes the code numbers, to extend the crack by one block. After some further iterations the new tip is examined and extended if necessary, and so on. Thus each crack will be extended to its compatible length, without requiring extra iterations. A simple method is also available to check that each crack is growing in the right direction, as explained in references 1 and 2.

5. REBAR.

5.1. From what has been explained it will be evident that the effect of bonded reinforcement can be calculated very simply, because the strain in each bar will be equal to the width of any crack which it intersects, divided by the distance between cracks. Hence the force in each bar can be calculated at each iteration, and then applied as loads tending to close the crack. This gives a close analogy to the actual situation in a reinforced concrete structure.

5.2. An examination of the formulae given in the Appendix will show that at each iteration an increment of deflection is calculated, and added to the previous total deflection, to provide the current value. This means that when deriving the force in the bar due to the strain, it is very easy

to include any non-linear stress-strain relations which may occur in the bar as it is extended. The same thing applies when considering the prestress force applied by each tendon in relation to its extension, as the structure deflects under load.

6. COMPARISON OF DYNAMIC RELAXATION WITH FINITE ELEMENTS.

6.1. After the foregoing brief outline of Dynamic Relaxation, as developed at the Institute of Atomic Energy, Sao Paulo, it will be helpful to make some comparison between it and the method of Finite Elements, as these are the two main alternative methods for the analysis of complex structures. This comparison is much simplified, and is intended to emphasise those aspects which are significant to the user of the programs.

6.2. The basic difference between the methods is that F.E. uses a big matrix representing the stiffness of the structure. In D.R. there are no algebraic matrices, but information concerning the stiffness is distributed throughout the structure in the various relations of each block with its neighbours, which are described by the array of code numbers. This means that if the structural stiffness changes, due to cracking or to yielding of materials, the F.E. matrix must be recalculated, at considerable cost in computer effort. There are ways of reducing this penalty with F.E. formulations, but they are not so easy, economical or accurate as with D.R., where it is only necessary to change some of the code numbers, which can be done even during the iterations.

6.3. As regards accuracy, it must be recognized that both methods are only approximate, in so far that the mathematical model is only a simplification of the actual structure, and the mathematical model itself cannot be solved with complete accuracy. In regard to both these points the accuracy of the F.E. solution will vary according to the type of finite element which is chosen, in relation to the actual structure. The D.R. method gives results which are at least as good as the best F.E. solution, other things being equal. However it must be recognised that other things are never equal and

the above comparison may have exceptions.

6.4. Economy of computer time and storage are important items. If it is required to examine an unchanging elastic structure, to which various loads are separately applied, then F.E. will show a saving in computer time, because once the stiffness matrix has been computed it can be used repeatedly. But in cases of non-linear structures, the problems outlined in paragraph 6.2. above will give a very decided advantage to D.R. as regards computer time and storage.

6.5. The authors assign high importance to convenience to the user, because they are anxious to place in his hands a simple and well understood tool for the analysis of all sorts of complex structures. In spite of the immense amount of effort put into F.E. programs, these remain a black box for the average user, which he can only employ by following the instruction manual slavishly. D.R. has the great advantage of being understandable in terms of the simple analogy already mentioned in paragraph 2.3. Then the necessary equations can be derived in the very simple manner shown in the Appendix. Only an elementary knowledge of computer programming is required to utilise D.R., because the program is just a series of nested DO loops, as also shown in the Appendix. Finally as the structure is represented by an array of blocks, it is only necessary to print out the stresses and deflections in arrays, when each will be in its correct relative position. This avoids the need to plot stress contours which is commonly required with F.E. calculations.

6.6. It will be obvious, from what has already been said about D.R., that these programs, originally written for static loads, are directly applicable to dynamic loads, in a manner which gives them some advantage over F.E. This is a matter basic to the subject of this paper, and it will be considered in more detail in the next section.

7. DYNAMIC LOADS.

7.1. The simple analogy, already mentioned in paragraph 2.3., shows that the D.R. programs are in fact assuming that the static loads are suddenly applied and then remain constant. The program then calculates the resulting stresses and deflections in each block, as the structure vibrates after the shock, under the influence of a near critical viscous damping. These values are not printed out, because they are of no interest, until the structure has come to rest, when we get the static answers for a static load. But it is obvious that we only need to transpose some of the output cards into the main DO loop if we want to know the values of the parameters at regular intervals during the iterations; see the program flow-chart in the Appendix. We can also change the damping factor to a value representative of the actual natural damping of the structure.

7.2. The simple changes of the last paragraph have provided a method of analysing the response of any structure to a suddenly applied loading, and all the previously discussed non-linearities, including cracking, can be included. If there is a loading transient varying over some time, it is only necessary to divide up the transient into the same time intervals as are being used in the program. Then the values of the loads in the various blocks can be changed, after the appropriate number of iterations, so that the changing loads are input correctly; this requires only some simple statements in the main DO loop.

7.3. It will be seen that in order to analyse the response of any structure, linear or non-linear, to general dynamic loads it is possible to use the basic programs, PV1, PV2, or PV3 according to the nature of the structure, with some minor changes. Moreover the one set of iterations can be used without interruption to calculate the response, and to input the loads, even when cracks are opening and closing. This is in contrast to F.E., where iterations within iterations are required. Experience has shown that in very complex cases, such as that to be described, it is advisable to control the

theoretical value.

9.5. After reflection, the elastic unloading wave will meet the plastic wave at a time of $3.636E-4$ secs, and the point where the waves meet is shown on the third diagram of figure 1. As a result of the meeting, new elastic waves will be propagated in both directions; but with the data given the particle velocity of these new waves will be zero, so there is no effect to the left of the meeting point, where stresses are zero. But the new elastic wave travelling to the right will act to unload the stresses in zone 1. It should just reach the impacted end of the bar after $4E-4$ secs, but it will be seen from the fourth diagram that in fact it is running a little late.

9.6. The new elastic unloading wave will be reflected from the impacted end with reversed sign, as shown in the fifth diagram, and reflected again from the free end, as shown in the sixth diagram. It will be seen that the wave is becoming dispersed, according to the computations. The computed deflections have not been shown in figure 1, but they are interesting, and may help to explain the discrepancy already remarked, that the plastic component of stress is under-computed.

9.7. As a result of the unbalanced stress system in the bar, during the early part of the transient, it appears that a general but not uniform translation towards the left is occurring, in addition to the wave motions. This is apparent from the changing pattern of deflections at the various time intervals, and from a print-out of velocities at $6E-4$ secs. But it was assumed, as is usual and as already mentioned, that unloading from the plastic zone followed the elastic modulus. That is to say for a given plastic strain, the loss of stress would be 100 times greater than the previous gain in stress. Hence it is possible to explain the difference between the computed and analytical plastic stresses. However it is not clear whether these discrepancies indicate some error in the computer, or in the admittedly approximate analysis and assumed stress-strain relations.

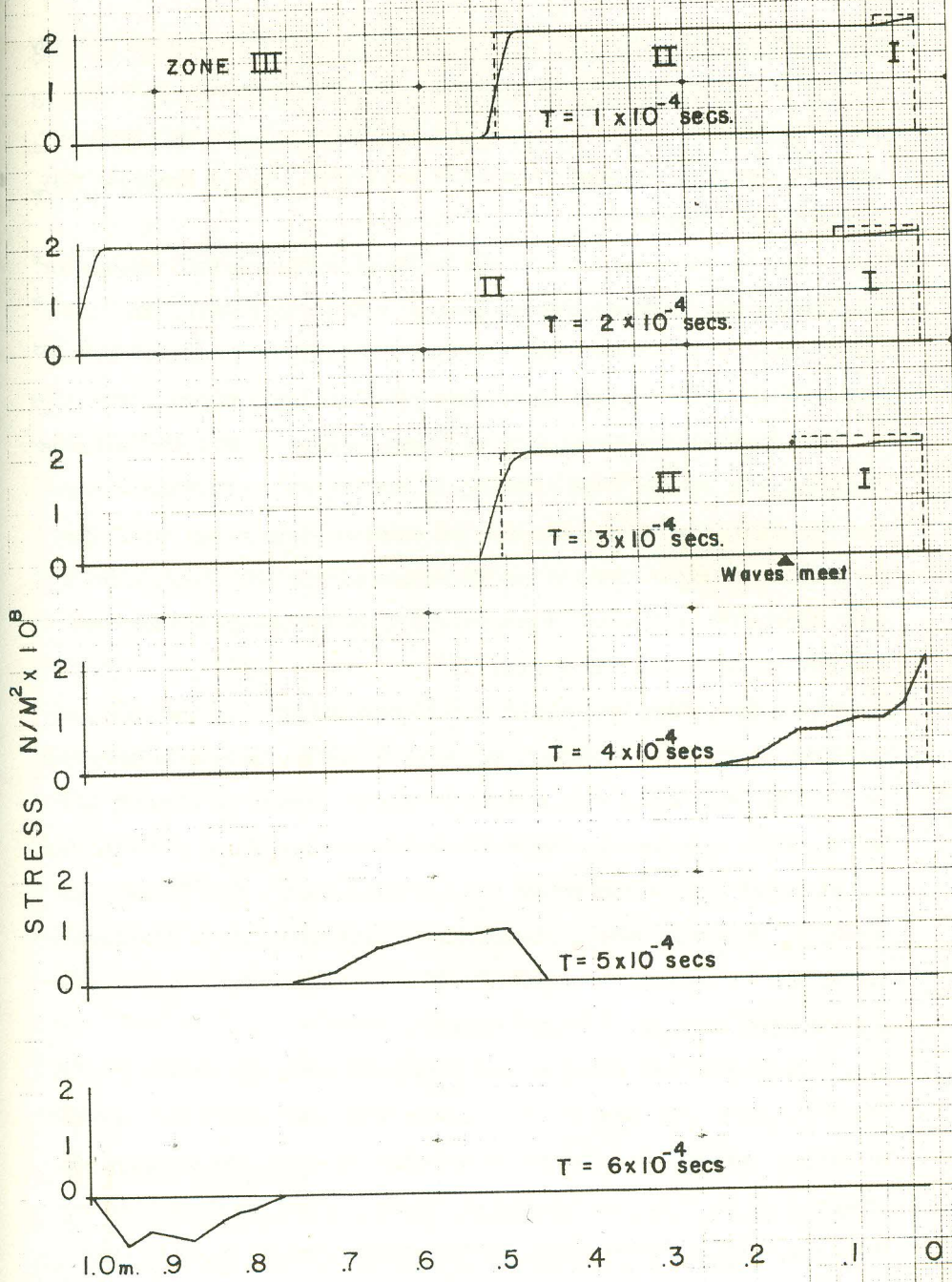


FIGURE I

could give results consistent with theory. As a start PVI was used to study a linearly elastic bar striking an anvil. The results agreed extremely well with classical theory, see addendum to reference 3, and were much more accurate than those obtained by Drittler and Gruner, using a specially written program, reference 4.

9.2. PVI was then used to study the impact of a finite-length uniform bar of elastic-linear strain-hardening material with a rigid anvil. The bar had a length of 1m and was travelling with a velocity of 10 m/sec.: it had a mass density of $8E3 \text{ kg/m}^3$ and Poissons ratio was taken as zero. The stress-strain curve was bi-linear, with an elastic modulus of $2E11 \text{ N/m}^2$ and plastic modulus of $2E9 \text{ N/m}^2$: the yield stress was $2E8 \text{ N/m}^2$, and it was assumed that unloading from the plastic zone was elastic. Damping was zero. For the purpose of computation the bar was represented by 200 blocks, with a time interval of $5E-7$ secs. A print-out of stresses and deflections was obtained at intervals of $1E-4$ secs.

9.3. The theoretical analysis of this problem was based on the treatment published by Professor W.Johnson, reference 5, and may be briefly described as follows. When impact with plastic deformation occurs, an elastic wave and a plastic wave are simultaneously propagated: they travel along the bar at velocities proportional to the roots of their moduli; i.e. in the case given, at $5E3$ and $5E2$ m/sec. respectively. There will then be three zones in the bar:- 1 has been traversed by both plastic and elastic waves, 2 by elastic waves only, and 3 is undisturbed.

9.4. The theoretical stress in zone 1 is $2.2E8 \text{ N/m}^2$, and in zone 2 it is $2E8 \text{ N/m}^2$. After $2E-4$ secs the elastic wave will have reached the far end of the bar, and will be reflected as an unloading wave. The foregoing are indicated on the first two diagrams of figure 1, where the theoretical results are shown by dashed lines, and the computed results by solid lines. It will be seen that there is excellent agreement for the elastic waves, but the plastic component of the total stress is computed at less than half its

onset of cracking by user intervention at intervals during the transient, because automation becomes very complicated. At each intervention all the parameters are stored on tape, so that the run can be recommenced without loss of computer time.

8. MODEL PCRV WITH INTERNAL EXPLOSION.

8.1. A recent paper, reference 3, described the use of PV2 to analyse the response of a model prestressed concrete containment vessel to an internal explosion. The cylindrical model had an external radius of 0.37m and a height of 0.725m; the cavity had a radius of 0.295m and height of 0.41m. It was prestressed by wrapping and by vertical tendons in the wall. The explosive was detonated under water contained in the model, and the resulting pattern of pressures on the cavity walls was calculated by a separate program: the pressures arose to a peak of about 50 MN/m^2 and fell again to zero within $4\text{E-}4$ secs.

8.2. The structure was represented in PV2 by 1,352 blocks, with a time interval of $2\text{E-}6$ secs. The calculations were continued for a real time of $1.5\text{E-}3$ secs, which required a computer running time of less than one hour, with a storage of 480K. The analysis was compared with results from an actual experiment on the model vessel, but unfortunately the theoretical pattern of pressure loads did not agree well with those which were measured in the experiment; there was also some uncertainty about the levels of prestress. However the crack pattern was well predicted, together with the interaction between various parts of the structure, but there was not good agreement as regards deflections.

9. DYNAMIC PLASTICITY.

9.1. Having completed the work described in the previous section, which included non-linearities due to cracking and to yielding of rebar, it was judged advisable to spend some time studying dynamic plasticity, using relatively simple configurations, in order to check whether Dynamic Relaxation

10. CONCLUSIONS.

10.1. The method of structural analysis known as dynamic relaxation has been described in essence, together with various additional features, producing a powerful, versatile and very simple means of analysing linear or non-linear structures, with static or dynamic loads. The authors are anxious to see these methods more widely employed, and will be glad to give any assistance which they can to enable others to write and use similar programs.

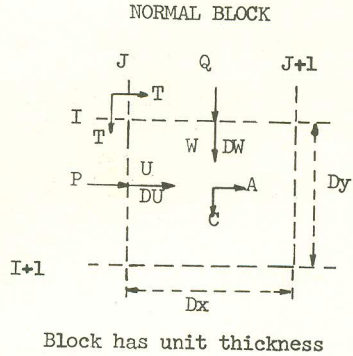
10.2. As illustrations of the capacity of dynamic relaxation in the field of non-linear structures subjected to dynamic loads, two problems have been described, one very complex with limited means of verification, and the other very much simpler, with a reasonably certain analytical solution.

APPENDIX.

BASIC DATA FOR PV1.

LIST OF VARIABLES.

- A Horizontal stress
- C Vertical stress
- K Damping coefficient
- DU Horizontal deflection
- DW Vertical deflection
- E Elastic modulus
- I, J Block coordinates
- P Applied horizontal pressure
- ν Poissons ratio
- Q Applied vertical pressure
- ρ Mass density
- T Shear stress
- Δt Time interval
- U Horizontal velocity
- W Vertical velocity
- Subscript_b Before iteration
- Subscript_a After iteration



FINITE DIFFERENCE STRESS EQUATIONS IN TERMS OF VELOCITIES

$$\sqrt{x} = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) \quad \text{for plane stress}$$

$$\therefore A = \frac{E}{1-\nu^2} \left[\frac{DU - DU(J+1)}{Dx} + \nu \frac{DW - DW(I+1)}{Dy} \right]$$

Differentiate with respect to time

$$\therefore \frac{DA}{Dt} = \frac{E}{1-\nu^2} \left[\frac{U - U(J+1)}{Dx} + \nu \frac{W - W(I+1)}{Dy} \right]$$

$$\text{But } DA = A_a - A_b$$

$$\therefore A_a = A_b + \frac{E \cdot Dt}{1-\nu^2} \left[\frac{U - U(J+1)}{Dx} + \nu \frac{W - W(I+1)}{Dy} \right]$$

$$\text{Similarly } C_a = C_b + \frac{E \cdot Dt}{1-\nu^2} \left[\frac{W - W(I+1)}{Dy} + \nu \frac{U - U(J+1)}{Dx} \right]$$

$$\text{Similarly } T_a = T_b + \frac{E \cdot Dt}{2(1+\nu)} \left[\frac{U(I-1) - U}{Dy} + \frac{W(J-1) - W}{Dx} \right]$$

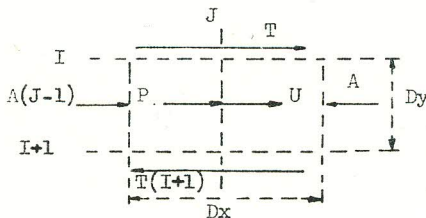
FINITE DIFFERENCE VELOCITY EQUATIONS IN TERMS OF STRESSES:

$P = Mf$ by Newton's Law

$$= \frac{M}{Dt} (dU + \text{viscous damping } kU)$$

$$Dy (A(J-1) + P - A) + Dx(T - T(I+1)) = \frac{r \cdot Dx \cdot Dy}{Dt} (dU + kU)$$

$$\text{But } dU = U_a - U_b \quad \text{and } U = \frac{U_a + U_b}{2}$$



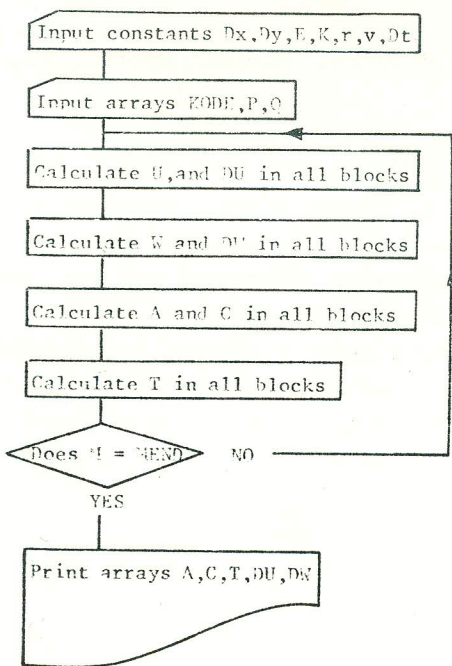
see figure above

$$U_a - U_b + \frac{k}{2} (U_a + U_b) = \frac{Dt}{r \cdot Dx \cdot Dy} \left[Dy(A(J-1) + Dx(T-T(I+1))) \right]$$

$$U_a = \frac{1-k/2}{1+k/2} U_b + \frac{Dt}{1+k/2} \times \frac{1}{r} \left[\frac{A(J-1)+P-A}{Dx} + \frac{T-T(I+1)}{Dy} \right]$$

Similarly $W_a = \frac{1-k/2}{1+k/2} W_b + \frac{Dt}{1+k/2} + \frac{1}{r} \left[\frac{C(I-1)+Q-C}{Dy} + \frac{T-T(J+1)}{Dx} \right]$

FLOW CHART OF PROGRAM PVI



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