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On the deterministic and stochastic solutions of Space Charge models and their impact on high resolution timing

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RPCs offer unique opportunities to investigate basic processes in gaseous electronics. The growth of a single avalanche can be studied in a regime where it reacts to its own field. This induces a saturation in its development, often described in a deterministic scenario by a nonlinear model. Once reinterpreted in a fully stochastic framework, the same feature corresponds to a negative feedback mechanism, which regulates the avalanche development and preserves its timing properties. Fluctuations are hence mostly produced in the initial phase of the growth. A clear evidence of the action of this stabilizing scheme is observed in data collected for single avalanches of fixed length.

1. Introduction

Experimentally, hints of the presence of a strong space charge induced saturation were discovered in trigger RPCs [1] and later studied in detail for timing RPCs [2]. Since the first attempts [3] to understand the physical basis of RPC operation in avalanche mode, it is well known that space charge is of primary importance to achieve high efficiencies with small gaps. The excellent timing resolutions of RPCs depend critically on a small avalanche growth timescale $1/(\alpha^\star v_d)$ [4–8], that results from very intense and highly uniform electric fields ($\approx$ 100 kV/cm). Fortunately, they can be tolerated with reasonable rate capabilities and with a low fraction of streamers, exclusively due to a strong reduction of the charge per avalanche by the space charge induced saturation.

Here the emphasis is on an analytic approach to the space charge mechanism in an attempt to shed light on the underlying physics. Three models have been proposed in the literature to account for this process: the original one by H.Raether [9], another by P.Fonte [10] and finally, more recently, one by G.Aielli et al. [11]. Even though, in principle, they are different and contain different physical hypotheses, once the respective free parameters have been constrained to the data, only small mutual deviations are predicted for the experimentally accessible observables, like average charge per avalanche as a function of applied voltage, fast vs. slow charge, ion current. This subject will be analyzed more thoroughly in a forthcoming publication. Here only the model of Ref. [10] will be dealt with, on account of its simpler analytical expressions. However, the qualitative conclusions are completely general and easily extended to the other cases.

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2. Deterministic and stochastic solutions

In all the models [9–11], the avalanche magnitude $\overline{m}$ has always been considered as a classical variable having a unique well defined value $\overline{m}(x)$ for each avalanche length x. For example, the model of Ref. [10] reads as

$$\frac{d\overline{m}}{dx} = \overline{m}\,\alpha_0\,\frac{m_{\mathrm{sat}}}{\overline{m} + m_{\mathrm{sat}}}\,, \tag{1}$$

where α_0 is the first Townsend coefficient in absence of any space charge influence and m_{sat} is the avalanche size at which saturation reduces α_0 to one half. In fact, Eq. (1) accounts for a situation where α decreases progressively when $\overline{m}$ gets closer to m_{sat}: there is no abrupt transition from absence to presence of a space charge effect, as opposed to the models of Ref.s [9,3]. However, once $\overline{m}$ is comparable to m_{sat}, the drop in α is very rapid [10]. The initial condition $\overline{m}(0) = 1$ corresponds to a single avalanche started by one electron. The analytic solution assumes the quite compact form

$$\overline{m}(x) = m_{\mathrm{sat}}\,W\left(\frac{1}{m_{\mathrm{sat}}}e^{1/m_{\mathrm{sat}}+\alpha_0\,x}\right)\,, \tag{2}$$

where $W(x)\ e^{W(x)} = x$ is the Lambert function [12]. Experimentally, $\overline{m}$ does not represent the induced, but rather the total produced charge for a single avalanche [10]. Eq. (2) exhibits a transition from an exponential behaviour, when $\overline{m}$ is below m_{sat}, to a linear one, when it is above. The same feature was first proposed as a direct footprint of space charge induced saturation, according to the logistic model, and corroborated with data in Ref. [11]. Eq.s (1,2) will be referred to as the deterministic interpretation of the model of Ref. [10].

In reality, avalanche growth is not a smooth process, but subject instead to a considerable amount of fluctuations, so that regarding the number of electrons as having a well defined value $\overline{m}$ for each length x is not fully appropriate. A better treatment can be devised by replacing the deterministic variable $\overline{m}$ with a random variable M that does not possess a single value for each length x but rather a probability distribution function (p.d.f.). Following a common practice in statistics, classical and random variables will be denoted with lower and upper cases respectively. Actually, W. Feller was the first [13] to realize the importance of this difference in the context of population dynamics introducing the stochastic version of the logistic equation, proposed in Ref. [11] as a suitable description also for the space charge action. Eq. (1) can then be reread into the new framework in two different ways, namely :

$$\frac{dM}{dx} = M\,\alpha_0\,\frac{m_{\mathrm{sat}}}{\langle M\rangle + m_{\mathrm{sat}}}\,, \tag{3}$$

where the average $\langle M\rangle$ is over different avalanches, or :

$$\frac{dM}{dx} = M\,\alpha_0\,\frac{m_{\mathrm{sat}}}{M + m_{\mathrm{sat}}}\,. \tag{4}$$

The former will be referred to as the 'average space charge model', while the latter simply as the 'fully stochastic model'. The main difference is that Eq. (3) is still linear in the random variable M, although with a non constant coefficient, while Eq. (4) is not. The physical meaning of such a distinction can be better appreciated picturing the development of an avalanche as a time discretized process. In Eq. (4), during saturation, if at one step the gain is bigger than the average, α for the next step is lower than the average, so that there is a higher chance for the growth as well to be less than the average. Correspondingly, the impact of the fluctuations, generated in each stage, onto the next is reduced. On the contrary, in Eq. (3), no means are available to damp the fluctuations, that will propagate their effect exponentially. In other words, the nonlinearity present at a deterministic level for describing the effect of saturation, when reinterpreted in a fully stochastic scenario, translates into a negative feedback mechanism stabilizing the avalanche dynamics. The possibility of such a self regulatory scheme was considered in Ref. [4], but due to the lack of any experimental evidence, it was neglected in the explicit treatment given there.

An analytic solution of Eq.s (3,4) can be constructed using the back extrapolation technique introduced in Ref. [10]. It is based on the observation that the avalanche size scale, at which fluctuations are generated ($M \lesssim 10^3$ electrons), and the one, at which the space charge has a relevant impact ($M \approx m_{\rm sat}$), are separated by at least two orders of magnitude. Hence, once in the region of importance for saturation, the growth process is completely smooth and the deterministic solution can be used to extrapolate back in time to an equivalent initial avalanche magnitude M_0 representing the total amount of all the fluctuations. Since the biggest contribution to the spread in final avalanche sizes is generated when the space charge does not exert a significant influence, it is natural to assume that M_0 will follow a Furry law and the final p.d.f. of M can be expressed as a change of random variable

$$\rho_M(m) = e^{-m_0(m)} \left| \frac{dm_0(m)}{dm} \right| , \tag{5}$$

leaving open only the nature of the relation between M and M_0. In absence of any space charge effect, it is obvious that $M_0 = M/\overline{m}(x)$ with $\overline{m}(x) = \exp(\alpha_0\, x)$ and a straightforward application of Eq. (5) leads to the usual result $\rho_M(m) = \exp(-m/\overline{m}(x))/\overline{m}(x)$. When the same *ansatz* $M_0 = M/\overline{m}(x)$ is kept and the expression for $\overline{m}(x)$ is replaced by Eq. (2), the result is again an exponential distribution with an average multiplication that now contains the saturation. This is the solution of the average space charge model Eq.(3). On the other hand, according to Ref. [10], if M_0 is, for example, larger than the average value, the avalanche will also saturate earlier. To take it into account, $m_{\rm sat}$ has been replaced by $m_{\rm sat}/M_0$ and $\overline{m}$ by M/M_0 in Eq. (2) and the resulting expression (or equivalent for m_0 and m) has been solved as a function of M_0 (or m_0), arriving at

$$M_0 = m_{\rm sat}\, W\left(\frac{M}{m_{\rm sat}} e^{\frac{M}{m_{\rm sat}} - \alpha_0\, x} \right) . \tag{6}$$

Inserting Eq. (6) into Eq. (5) gives the final result

$$\rho_M(m) = e^{-m_0(m)} \frac{1 + \frac{m_{\rm sat}}{m}}{1 + \frac{m_{\rm sat}}{m_0(m)}} . \tag{7}$$

Eq. (7) agrees with the numerical solution of Eq. (4) to better than 1% for typical values of $m_{\rm sat} \approx 5 \cdot 10^5$, the maximum deviation being observed in the region below $\langle M \rangle$. The accuracy improves as $m_{\rm sat}$ increases. The most interesting feature of Eq. (7) is that it has an exponential shape only in the limit of $\langle M \rangle \ll m_{\rm sat}$, i.e. when the space charge influence is negligible (note: the convergence is not uniform). On the contrary, for $\langle M \rangle \approx m_{\rm sat}$, a maximum appears in ρ_M indicating a suppression of the fluctuations by the negative feedback. Actually, probing the p.d.f deeper into the space charge affected region, renders the peak sharper. A second point is that, as discovered by W.Feller [13], for a nonlinear stochastic differential equation, like Eq. (4), the average of the stochastic solution $\langle M \rangle$ does not coincide with the deterministic solution Eq. (2). In the present case, it is always $\langle M \rangle < \overline{m}$ and a difference up to 15% in the estimate of $m_{\rm sat}$ can be found, depending on which one of the two is fitted to the data. This is not the case for a linear stochastic differential equation, like Eq. (3), as is evident also from the constructed solution.

From a mathematical point of view, both Eq. (3) and Eq. (4) are legitimate extensions of Eq. (1) to the case of a stochastic avalanche size M. The question arises as to which one is closer to reality. New data were collected with a dedicated setup evolved from Ref. [14]. In particular, a quartz fibre was used to shine U.V. light into the gap and initiate all avalanches from the cathode by extracting a photoelectron. By reducing the U.V. light intensity, it was possible to work in a single avalanche regime. The high accuracy in the mechanical construction of the 0.3 mm gap ensured that the length of the avalanche was fixed and known. The large bandwidth electronics was located into the gas box, directly onto the chamber, whose dimensions were optimised to match the input impedance of the preamplifier stage. The standard timing-RPC gas mixture was used [6]. The total charge [10] was calibrated to yield the multiplication M in number of electrons. A pulse height spectrum accumulated at a bias voltage of 2.6 kV is reported, after normalization, in Fig. 1. Under such operating conditions, its shape intimately reflects the avalanche dynamics, all the rest being constant. Once the value of $m_{\rm sat} = 6 - 7 \cdot 10^6$ is extracted by fitting the ex-

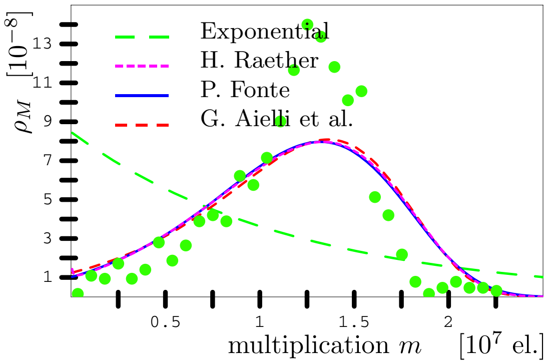

Figure 1. Measured pulse height spectrum for single avalanches of 0.3 mm length. The gas mixture is the standard one and the bias voltage 2.6 kV. The solutions of Eq.s (3,4) are superimposed for comparison. The stochastic solution of the models of Ref.s [9,11] are also displayed.

perimental dependence of $\langle M \rangle$ from the applied voltage, no more degrees of freedom are left and the described solutions of Eq.s (3,4) can be compared to the data (see Fig. 1, widest pitch dashed and continuous curves, respectively). The same procedure, employed to derive Eq. (7) and the necessary parameters, have been extended to the models of Ref.s [9,11] and the results are also displayed in Fig. 1. Clearly, the compression of the spectrum of the fluctuations by the negative feedback is present in the data, which are seen to be reasonably reproduced only by the fully stochastic scenario. On the other hand, the different descriptions of the saturation properties all yield very close predictions and can not be experimentally discriminated. More details will be given in a future work.

3. Single avalanche timing

A lack of influence of the space charge effect on timing properties of RPCs has been found in Monte-Carlo simulations [15]. It is then interesting to study the same problem within the previous models of avalanche dynamics. A simple argument linking the multiplication and the time domains has been illustrated in Ref. [5], while a more compact one was introduced in Ref. [4]. Here it should just be recalled that they both regard as ideal the timing threshold m_t on the avalanche magnitude or, equivalently, on the induced current [6]. To streamline the discussion, the drift velocity v_d is assumed to be saturated in the range of electric field strengths relevant to an important space charge effect, so that a non dimensional quantity $\tau = s_0\, t$ with $s_0 = \alpha_0\, v_d$ can be defined. All the results can be re-interpreted again as a change of variable, like in Eq. (5), but now from the exponential distribution in M_0 to $\mathcal{T} = s_0\, T$.

In absence of space charge, the p.d.f in $\mathcal{T}$ is

$$\begin{cases} \rho_{\mathcal{T}}(\tau) = e^{-m_0(\tau)}\, m_0(\tau) \\ m_0(\tau) = \dfrac{m_t}{e^{\tau}} \end{cases} , \qquad (8)$$

in agreement with Ref.s [4,5]. The time at maximum is $\tau_{\max} = \ln(m_t)$, the average time is $\langle \tau \rangle = \ln(m_t) + C$, with C the Euler-Mascheroni constant, and the variance is $\langle (\tau - \langle \tau \rangle)^2 \rangle = \pi^2/6$ [5]. If $\rho_{\mathcal{T}}$ is rewritten as a function of $\tau - \tau_{\max}$, then it does not depend on m_t itself [5,6].

When the effect of the space charge is considered within Eq. (3), the p.d.f. in τ is

$$\begin{cases} \rho_{\mathcal{T}}(\tau) = e^{-m_0(\tau)} \dfrac{m_0(\tau)}{1 + \frac{m_t}{m_{\text{sat}}}\, m_0(\tau)} \\ m_0(\tau) = \dfrac{m_t}{m_{\text{sat}}\, W\left(\frac{1}{m_{\text{sat}}} e^{1/m_{\text{sat}} + \tau}\right)} \end{cases} . \qquad (9)$$

The same result can be obtained by applying directly Eq. (26) of Ref. [4], valid only once the negative feedback mechanism is neglected, to Eq. (2) where $\alpha_0\, x$ is to be replaced by τ. The time at maximum is now given by

$$\begin{cases} \tau_{\max} = \ln \dfrac{w_{\max}\, e^{w_{\max}}}{\frac{1}{m_{\text{sat}}}\, e^{1/m_{\text{sat}}}} \\ w_{\max} = \dfrac{1}{4}\sqrt{\left(1 - \dfrac{m_t}{m_{\text{sat}}}\right)^2 + 8\dfrac{m_t}{m_{\text{sat}}}} \\ \qquad - \dfrac{1}{4}\left(1 - \dfrac{m_t}{m_{\text{sat}}}\right) \end{cases} . \qquad (10)$$

The limit for $m_{\text{sat}} \to \infty$ of Eq. (9) and Eq. (10) are Eq. (8) and $\ln(m_t)$, respectively. The important property of Eq. (9) to be noted is that, if re-expressed in terms of $\tau - \tau_{\max}$, it becomes a function of $m_t/m_{\rm sat}$ and not of each separately. Unfortunately, Eq. (9) is characterized by a very strong tail decreasing only linearly, rendering all moments undefined. Nevertheless, it is not difficult to realize that as $m_t/m_{\rm sat}$ increases, i.e. the threshold is placed deeper and deeper in the saturation region, the timing properties are gradually lost.

If the framework of Eq.(4) is adopted, the p.d.f in τ is

$$\begin{cases} \rho_{\mathcal{T}}(\tau) = e^{-m_0(\tau)} \dfrac{m_0(\tau)}{1+\frac{m_0(\tau)}{m_{\rm sat}}} \\ m_0(\tau) = m_{\rm sat}\, W\left(\dfrac{m_t}{m_{\rm sat}} e^{\frac{m_t}{m_{\rm sat}}-\tau}\right) \end{cases} . \quad (11)$$

The time at maximum is then provided by

$$\begin{cases} \tau_{\max} = \ln \dfrac{\frac{m_t}{m_{\rm sat}}\, e^{m_t/m_{\rm sat}}}{w_{\max}\, e^{w_{\max}}} \\ w_{\max} = \sqrt{\dfrac{1}{4}+\dfrac{1}{m_{\rm sat}}} - \dfrac{1}{2} \end{cases} . \quad (12)$$

Again the limit for $m_{\rm sat} \to \infty$ of Eq. (11) and Eq. (12) are Eq. (8) and $\ln(m_t)$, respectively. If Eq. (11) is transformed into a function of $\tau - \tau_{\max}$, only a dependence from $m_{\rm sat}$ is left. It is then obvious that the timing properties are preserved, no matter where m_t is located. It is also possible to explicitly calculate $\langle\tau\rangle = \ln m_t + C + (m_t - 1)/m_{\rm sat}$ and $\langle(\tau - \langle\tau\rangle)^2\rangle = \pi^2/6 + (2 + 1/m_{\rm sat})/m_{\rm sat}$. The correction terms containing $m_{\rm sat}$ are very small, under ordinary conditions (where Eq. (7) is still a reasonable approximation of the solution of Eq. (4)), and should not be taken too seriously. If they are neglected, the same values of Eq. (8) are found. This supports the idea that all the time fluctuations are generated in the first steps of the avalanche development and that they are not allowed to grow, during the saturation phase, by the negative feedback contained in Eq. (4) but not in Eq. (3). Such a stabilizing action regards not only the growth itself, but very likely also the electric field resulting from the screening action of the space charge, so that the last conclusion remains unaltered, even though v_d is not saturated. Actually, this was the case in the MC of Ref. [15].

4. Conclusions

A successful interpretation of the shape of the size spectrum for fixed-length-single avalanches requires a stochastic scenario. The observed compression of the dynamics by the space charge effect can be reproduced only if the non-linearity, necessary to describe the saturation, is allowed to fully manifest in the stochastic framework as a negative feedback mechanism. The same stabilizing action preserves the timing properties so that the vast majority of the fluctuations are generated in the initial stage. If confirmed, this would provide a completely general basis for the approach followed in Ref.s [4–8] to calculate the time response of an RPC.

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REFERENCES

1. R. Cardarelli et al., NIMA 382 (1996) 470.
2. P. Fonte et al., NIMA 477 (2002) 17.
3. M. Abbrescia et al., NPB (Proc. Supl.) 78 (1999) 459.
4. A. Mangiarotti et al., NIMA 482 (2002) 192.
5. A. Gobbi et al., NIMA 508 (2003) 23.
6. A. Blanco et al., NIMA 513 (2003) 8.
7. A. Mangiarotti et al., NIMA 533 (2004) 16.
8. A. Blanco et al., NIMA 535 (2004) 272.
9. H. Raether, Electron Avalanches and Breakdown in Gases, Butterworths 1964.
10. P. Fonte, NIMA 456 (2000) 6.
11. G. Aielli et al., NIMA 508 (2003) 6.
12. R.M. Corless et al., Adv. Comp. Math., 5 (1996) 329.
13. W. Feller, Acta Biotheoretica 5 (1939) 1.
14. P. Fonte et al., NIMA 433 (1999) 513
15. C Lippmann et al., NIMA 517 (2004) 54.